University of Helsinki

Department of Mathematics

Solving the Quintic

—

From Tschirnhaus' Transformations to Klein's Icosahedral Equation

Seminar paper

of

Christian Weiß

Seminar: Historiallinen johdatus Galois'n teoriaan

Lecturer: FT Jukka Pihko

December 2002

Preface

The history of algebra is the history of solving equations. The term *algebra* comes from the Latin word *algebra* which was itself built out of the Arabic word *al-ǧabr*. *al-ǧabr* originally meant 'restauration' or 'recovery', but in the mathematical sense it was used for the transformation of a negative value into a positive one by moving it on the other side of the equation. This word is contained in the title *ilm al-ǧabr wa l-muqâbala* ('science of recovering and comparing') of the Persian-Arabian mathematician Al-Ḫwārizmī, written about the year 825 A.D. The second term in this title, 'l-muqâbala', meant simplifying the equation by comparing both sides and removing terms from them.
But the Arabs were not the first who tried (succesfully) to solve equations, even the Babylonians (1000 B.C.) and the Greeks (300 B.C.) developped methods of solving (quadratic) equations.

But after the Greeks it took more than 1500 years until new developments in solving equations have been made in Europe. It was about 1515 when Scipione del Ferro solved a cubic equation of the form $x^3+mx=n$ (but never published it), and the first complete solution of cubics was given in 1539 by Girolamo Cardan in his book 'Artis Magnæ– Sive De Regulis Algebraicis' ('The Great Art – or the Rules of Algebra'), which also includes a solution of the quartic equation, found by his student Ludovico Ferrari.
One may expect that after this rapid development it was only a question of time when solutions of higher degree equations will be found, but the next markable step was Newtons numerical approximation of the roots of a function in 1669. A few years later, E. W. von Tschirnhausen developped a method of removing at least the $(n-1)^{\text{th}}$ and $(n-2)^{\text{th}}$ term of a polynomial equation and transforming it by that way in a simpler form.

Gauß,
Johann Carl Friedrich,
** 30.04.1777*
in Brunswick/Germany
† 23.02.1855
in Göttingen/Germany

These transformations mark the beginning of this seminar paper. The idea of the Tschirnhaus transformations were used afterwards by Bezout and Euler (about 1762), too, and in 1786 Bring found a way of transforming the general fifth order equation in the form $x^5 + bx + c$, which will also be discussed in this paper.

13 years later, Carl Friedrich Gauß proved the so-called *fundamental theorem of algebra*, which says that every non-constant real polynomial has at least one root in $\mathbb{C}$, or equivalently, that the number of linear factors of a non-zero polynomial over $\mathbb{C}$ is equal to its degree. Then came the year 1826, the year when Abel published the proof of one of his most famous results:
The general n-th order polynomial equation is for $n \geq 5$ not solvable by radicals!
So it is impossible to solve polynomial equations of degree higher than four? No! It is not possible by using the 'standard' operations and radicals (i.e. roots) only! And a way of solving such higher order equations can be found by Felix Klein, who solved the quintic equation and showed a way of using his ideas for equations of degree ≥ 6. Klein's method of solving the quintic will be the end of this seminar paper.

Abel,
Niels Henrik,
** 05.08.1802*
in Frindoe/Norway
† 06.04.1829
in Froland/Norway

This seminar paper originated in the course of the seminar "Historiallinen johdatus Galois'n teoriaan", taught by FT Jukka Pihko, and was based on the book "Galois' Theory of Algebraic Equations", written by Jean-Pierre Tignol [8], although it was only used (among others) for the sections 1.2 and 2.1.

The historical pictures used in this seminar paper are all taken from [11], except the print of Tschirnhaus' article on page 7, this image can be found in [12].

Helsinki, December 2002 Christian Weiß

Contents

Chapter 1

The Tschirnhaus Transformations

1.1 The Resultant of two Polynomials

Before we begin to talk about the *resultant* of two polynomials we will first repeat some basic facts about the division of polynomials and a result of linear algebra of solving linear equations.

1.1.1 Definition Assume $\mathbb{F}$ to be an arbitrary field, then $\mathbb{F}[x]$ is the set of all polynomials (German: Polynome) $p(x) = a_n x^n + ... + a_1 x + a_0$ with coefficients (German: Koeffizienten) $a_n, ..., a_0 \in \mathbb{F}$.
The *degree* (German: Grad) $deg(p)$ is then defined to be the greatest integer $n \in \mathbb{N}_0$ such that $a_n \neq 0$ and $a_k = 0$ for $k \geq n+1$.
The corresponding coefficient is called the *leading coefficient* (German: Leitkoeffizient), and if this leading coefficient equals 1, then the polynomial is called a *monic polynomial* (German: normiertes Polynom).

It can be shown that $\mathbb{F}[x]$ is a ring.

1.1.2 Theorem (Euclidean Division Property) Let $p_1, p_2 \in \mathbb{F}[x]$ be two arbitrary polynomials with $p_2 \neq 0$. Then there exist two polynomials $q(x), r(x) \in \mathbb{F}[x]$ such that

$$p_1 = p_2 \cdot q + r \qquad \text{and} \qquad deg(r) < deg(p_2).$$

The polynomials are uniquely determined by these properties.

Proof: See Tignol [8], p. 43f. #

1.1.3 Definition Let $p_1, p_2 \in \mathbb{F}[x]$ be two arbitrary polynomials. p_2 *divides* p_1 (German: teilt) if there exists a polynomial $q \in \mathbb{F}[x]$ such that $p_1 = q \cdot p_2$.
A polynomial $p \in \mathbb{F}[x]$ is said to be *irreducible* (German: irreduzibel) in $\mathbb{F}[x]$ if $deg(p) > 0$ and p is not divisible by any polynomial $q \in \mathbb{F}[x]$ with $0 < deg(q) < deg(p)$.
A *greatest common divisor (GCD)* of p_1 and p_2 (German: größter gemeinsamer Teiler

$ggT(p_1, p_2)$) is a polynomial $d \in \mathbb{F}[x]$ with the following properties:
(i) d divides both p_1 and p_2;
(ii) if $s \in \mathbb{F}[x]$ is a polynomial which also divides p_1 and p_2, then s divides d.
If 1 is a *GCD* of p_1 and p_2, then p_1 and p_2 are *relatively prime* (German: teilerfremd).

Finally we repeat the following lemma from linear algebra:

1.1.4 Lemma Suppose an arbitrary field $\mathbb{F}$, a matrix $A \in \mathbb{F}^{n \times n}$ and a vector $b \in \mathbb{F}^n$ with $n \in \mathbb{N}$. If the linear system of equations $Ax = b$ with $x \in \mathbb{F}^n$ is solvable, then the following properties are equivalent:
(i) $Ax = b$ has a unique solution.
(ii) $det(A) \neq 0$.

Proof: A proof can be found in nearly every book over linear algebra, for example in M. Menth: Lineare Algebra, Würzburg 1997. #

1.1.5 Definition The *resultant* $Res(p, q)$ (German: Resultante) of two polynomials $p(x) = a_n x^n + ... + a_0,\ q(x) = b_m x^m + ... + b_0 \in \mathbb{F}[x]$ with $n, m \in \mathbb{N}$ is the determinant defined by

$$Res(p,q) := \left| \begin{array}{cccc|cccc} a_n & & & & b_m & & & \\ a_{n-1} & a_n & & & b_{m-1} & b_m & & \\ a_{n-2} & a_{n-1} & \ddots & & b_{m-2} & b_{m-1} & \ddots & \\ \vdots & \vdots & \ddots & a_n & \vdots & \vdots & \ddots & b_m \\ a_0 & a_1 & & a_{n-1} & b_0 & b_1 & & b_{m-1} \\ & a_0 & & \vdots & & b_0 & & \vdots \\ & & \ddots & a_1 & & & \ddots & b_1 \\ & & & a_0 & & & & b_0 \\ \multicolumn{4}{c|}{\underbrace{\hspace{6em}}_{m \textbf{ columns}}} & \multicolumn{4}{c}{\underbrace{\hspace{6em}}_{n \textbf{ columns}}} \end{array} \right|,$$

which is an $(m + n) \times (m + n)$-determinant.

Our aim in the following discussion is to show which kind of information about two polynomials can be obtained by calculating their resultant. For this purpose we show first the following

1.1.6 Lemma Suppose $p, q \in \mathbb{F}[x]$ arbitrary.
p and q have a non-constant common divisor $\varphi \in \mathbb{F}[x]$ $\Leftrightarrow$ there exists $h_1, h_2 \in \mathbb{F}[x]$ with

$$h_1 \cdot p = h_2 \cdot q \qquad (\star),$$

$deg(h_1) \leq deg(q) - 1$, $deg(h_2) \leq deg(p) - 1$ and not both $h_1, h_2 \equiv 0$.

Proof:
"$\Rightarrow$": If φ is a non-constant common divisor of p and q, then there exist by definition polynomials $h_1, h_2 \in \mathbb{F}[x]$ with

$$p = \varphi \cdot h_2 \qquad \text{and} \qquad q = \varphi \cdot h_1,$$

so $(\star)$ is fulfilled.
"$\Leftarrow$": Suppose that $(\star)$ is given. Then both sides of the equation can be split in a product of irreducible polynomials. All such factors of p must be found on the right side, too, and as $deg(h_2) \leq deg(p) - 1$, at least one of the irreducible factors of p must appear among the factors of q. So there exists a common divisor. #

1.1.7 Theorem Suppose p and q given like in definition 1.1.5 with leading coefficients $a_n, b_m \neq 0$, then the following two properties are equivalent:
(i) $Res(p,q) = 0$.
(ii) p and q have a non-constant common divisor.

Proof: By lemma 1.1.6, (ii) is equivalent to the existence of two polynomials

$$h_1(x) = c_{m-1}x^{m-1} + ... + c_0 \qquad \text{and} \qquad h_2(x) = -d_{n-1}x^{n-1} - ... - d_0$$

with $h_1 \cdot p = h_2 \cdot q$. Calculating the products on both sides of the equation and comparing the coefficients yields the following system of linear equations:

$$\begin{array}{ccccccc} a_n c_{m-1} & & & = & -b_m d_{n-1} & & \\ a_{n-1}c_{m-1}+ & a_n c_{m-2} & & = & -b_{m-1}d_{n-1}- & b_m d_{n-2} & \\ & \cdots & & = & & \cdots & \\ & a_0 c_1 + & a_1 c_0 & = & & -b_0 d_1 - & b_1 d_0 \\ & & a_0 c_0 & = & & & -b_0 d_0 \end{array} .$$

An equivalent formulation for this system is given by

$$\left(\begin{array}{cccc|cccc} a_n & & & & b_m & & & \\ a_{n-1} & a_n & & & b_{m-1} & b_m & & \\ a_{n-2} & a_{n-1} & \ddots & & b_{m-2} & b_{m-1} & \ddots & \\ \vdots & \vdots & \ddots & a_n & \vdots & \vdots & \ddots & b_m \\ a_0 & a_1 & & a_{n-1} & b_0 & b_1 & & b_{m-1} \\ & a_0 & & \vdots & & b_0 & & \vdots \\ & & \ddots & a_1 & & & \ddots & b_1 \\ & & & a_0 & & & & b_0 \end{array}\right) \cdot \begin{pmatrix} c_{m-1} \\ c_{m-2} \\ \vdots \\ c_1 \\ c_0 \\ d_{n-1} \\ \vdots \\ d_1 \\ d_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

which is solvable, since it is possible to choose all $c_{m-1} = ... = d_0 = 0$. But (ii) and 1.1.6 make it necessary that there is a non-trivial solution which exists by Lemma 1.1.4 if and only if the determinant of the matrix (and that is the resultant!) vanishes. #

1.1.8 Example The resultant of two polynomials can be interpreted as an indicator for common divisors. We will demonstrate this at the following example, using the computer algebra system mathematica 4.0 for the calculations:

Define $p(x) = x^5 - \frac{23}{4}x^4 - \frac{15}{2}x^3 + \frac{17}{24}x^2 + \frac{37}{4}x + 3$ and

$q(x) = x^6 + \frac{163}{40}x^5 - \frac{97}{5}x^4 - \frac{2009}{40}x^3 + \frac{2597}{20}x^2 + \frac{206}{5}x - 84$, then the resultant $Res(p,q) = 0$:

$$\texttt{Resultant}\left[3 + \frac{37\,x}{4} + \frac{17\,x^2}{24} - \frac{15\,x^3}{2} - \frac{23\,x^4}{24} + x^5,\right.$$
$$\left.-84 + \frac{206\,x}{5} + \frac{2597\,x^2}{20} - \frac{2009\,x^3}{40} - \frac{97\,x^4}{5} + \frac{163\,x^5}{40} + x^6,\ x\right]$$

0

So we know that both polynomials have a common divisor and we should calculate the greatest common divisor of them:

$$\texttt{Factor}\left[\texttt{PolynomialGCD}\left[3 + \frac{37\,x}{4} + \frac{17\,x^2}{24} - \frac{15\,x^3}{2} - \frac{23\,x^4}{24} + x^5,\right.\right.$$
$$\left.\left.-84 + \frac{206\,x}{5} + \frac{2597\,x^2}{20} - \frac{2009\,x^3}{40} - \frac{97\,x^4}{5} + \frac{163\,x^5}{40} + x^6\right]\right]$$

$$\frac{1}{120}\,(-3 + x)$$

The (monic) common factor of p and q is therefore $(x - 3)$.

In the next section we will illustrate the use of the resultant by simplifying polynomials in the sense that we look for transformations of the variable to remove some terms in the polynomial; the transformations we are describing there will be the *Tschirnhaus transformations*.
But before doing this we will proof two lemmas which we can use in the next section:

1.1.9 Lemma Suppose $n \in \mathbb{N}$ and $p(x) := x^n + a_{n-1}x^{n-1} + ... + a_0$ and $q(x) = x + b_0$ in $\mathbb{F}[x]$. Then

$$Res(p,q) = b_0^n + \sum_{k=0}^{n-1} a_k(-1)^{n-k}b_0^k.$$

Proof: The proof is done by induction. For $n = 1$ we have

$$Res(x + a_0, x + b_0) = \begin{vmatrix} 1 & 1 \\ a_0 & b_0 \end{vmatrix} = b_0 - a_0 = b_0^1 + \sum_{k=0}^{0} a_k(-1)^{1-k}b_0^k.$$

Now assume we have shown the above formula for $\leq n-1$, then look at the case n:

$$Res(p,q) = \left.\begin{vmatrix} 1 & 1 & & & & \\ a_{n-1} & b_0 & 1 & & & \\ a_{n-2} & & b_0 & \ddots & & \\ \vdots & & & \ddots & 1 & \\ a_1 & & & & b_0 & 1 \\ a_0 & & & & & b_0 \end{vmatrix}\right\} n+1 \text{ rows}$$

$$\overset{\text{developping}}{\underset{\text{last row}}{=}} (-1)^n a_0 + b_0 \begin{vmatrix} 1 & 1 & & & \\ a_{n-1} & b_0 & 1 & & \\ a_{n-2} & & b_0 & \ddots & \\ \vdots & & & \ddots & 1 \\ a_1 & & & & b_0 \end{vmatrix} \overset{\text{induction}}{\underset{\text{hypothesis}}{=}} b_0(b_0^{n-1} + \sum_{k=0}^{n-2} a_{k+1}(-1)^{n-1-k} b_0^k) + (-1)^n a_0$$

$$= b_0^n + \sum_{k=0}^{n-2} a_{k+1}(-1)^{n-(k+1)} b_0^{k+1} + (-1)^{n-0} a_0 b_0^0 = b_0^n + \sum_{k=0}^{n-1} a_k (-1)^{n-k} b_0^k.$$

So the proof is complete. #

1.1.10 Lemma Suppose $n \in \mathbb{N}$ and $p(x) := a_n x^n + a_{n-1}x^{n-1} + ... + a_0$ of degree n and $q(x) = x^2 + b_1 x + b_0$ in $\mathbb{F}[x]$. Then for $n \geq 3$
(i) $Res(p,q) = Res(\ (a_{n-1} - a_n b_1)x^{n-1} + (a_{n-2} - a_n b_0)x^{n-2} + a_{n-3}x^{n-3} + ... + a_0, \quad q(x))$.
(ii) $Res(p,q) = a_n^2 b_0^n + A_{n-1} b_0^{n-1} + A_{n-2} b_0^{n-2} + P_{n-3}(b_0)$
with $A_{n-1} = a_{n-1}^2 - 2a_n a_{n-2} - a_n a_{n-1} b_1$,
$A_{n-2} = a_{n-2}^2 - 2a_{n-1}a_{n-3} + 2a_n a_{n-4} - a_{n-1}a_{n-2}b_1 + 3a_n a_{n-3} b_1 + a_n a_{n-2} b_1^2$
and $P_{n-3}(b_0)$ a polynomial in b_0 of degree $\leq n-3$; for $n \geq 4$.
(iii) For $a_n = 1$ we have

$$Res(p,q) = b_0^n + A_{n-1} b_0^{n-1} + A_{n-2} b_0^{n-2} + P_{n-3}(b_0)$$

with $A_{n-1} = a_{n-1}^2 - 2a_{n-2} - a_{n-1} b_1$,
$A_{n-2} = a_{n-2}^2 - 2a_{n-1}a_{n-3} + 2a_{n-4} - a_{n-1}a_{n-2}b_1 + 3a_{n-3}b_1 + a_{n-2}b_1^2$
and $P_{n-3}(b_0)$ a polynomial in b_0 of degree $\leq n-3$; for $n \geq 4$.

Proof: (i)

$$Res(p,q) = \begin{vmatrix} a_n & & 1 & & & & \\ a_{n-1} & a_n & b_1 & 1 & & & \\ a_{n-2} & a_{n-1} & b_0 & b_1 & \ddots & & \\ \vdots & \vdots & & b_0 & \ddots & 1 & \\ a_1 & a_2 & & & \ddots & b_1 & 1 \\ a_0 & a_1 & & & & b_0 & b_1 \\ & a_0 & & & & & b_0 \end{vmatrix} = \begin{vmatrix} 0 & & 1 & & & & \\ a_{n-1} - a_n b_1 & a_n & b_1 & 1 & & & \\ a_{n-2} - a_n b_0 & a_{n-1} & b_0 & b_1 & \ddots & & \\ \vdots & \vdots & & b_0 & \ddots & 1 & \\ a_1 & a_2 & & & \ddots & b_1 & 1 \\ a_0 & a_1 & & & & b_0 & b_1 \\ & a_0 & & & & & b_0 \end{vmatrix}$$

$$= \begin{vmatrix} a_{n-1}-a_nb_1 & a_n & 1 & & & \\ a_{n-2}-a_nb_0 & a_{n-1} & b_1 & \ddots & & \\ \vdots & a_{n-2} & b_0 & \ddots & 1 & \\ a_1 & \vdots & & \ddots & b_1 & 1 \\ a_0 & a_1 & & & b_0 & b_1 \\ & a_0 & & & & b_0 \end{vmatrix} = \begin{vmatrix} a_{n-1}-a_nb_1 & 0 & 1 & & & \\ a_{n-2}-a_nb_0 & a_{n-1}-a_nb_1 & b_1 & \ddots & & \\ a_{n-3} & a_{n-2}-a_nb_0 & b_0 & \ddots & 1 & \\ \vdots & a_{n-3} & & \ddots & b_1 & 1 \\ a_0 & \vdots & & & b_0 & b_1 \\ & a_0 & & & & b_0 \end{vmatrix}$$

$$= Res(\ (a_{n-1}-a_nb_1)x^{n-1} + (a_{n-2}-a_nb_0)x^{n-2} + a_{n-3}x^{n-3} + \ldots + a_0,\ x^2+b_1x+b_0).$$

(ii) For proofing this formula we use again induction and start with $n = 4$:

```
Collect[Resultant[a4 x^4 + a3 x^3 + a2 x^2 + a1 x + a0, x^2 + b1 x + b0, x], b0]

a0^2 + a4^2 b0^4 - a0 a1 b1 + a0 a2 b1^2 - a0 a3 b1^3 + a0 a4 b1^4 + b0^3 (a3^2 - 2 a2 a4 - a3 a4 b1) +
  b0^2 (a2^2 - 2 a1 a3 + 2 a0 a4 - a2 a3 b1 + 3 a1 a4 b1 + a2 a4 b1^2) +
  b0 (a1^2 - 2 a0 a2 - a1 a2 b1 + 3 a0 a3 b1 + a1 a3 b1^2 - 4 a0 a4 b1^2 - a1 a4 b1^3)
```

It is easy to verify that the coefficients of b_0^3 and b_0^2 are of the correct form.
Now suppose we have shown the above formula for $\leq n-1$, then look at the case n:

$$Res(p,q) \overset{\text{(i)}}{=} Res(\ (a_{n-1}-a_nb_1)x^{n-1} + (a_{n-2}-a_nb_0)x^{n-2} + a_{n-3}x^{n-3} + \ldots + a_0,\ x^2+b_1x+b_0)$$

$$\underset{\text{hypothesis}}{\overset{\text{induction}}{=}} (a_{n-1}-a_nb_1)^2b_0^{n-1}$$
$$+ ((a_{n-2}-a_nb_0)^2 - 2(a_{n-1}-a_nb_1)a_{n-3} - (a_{n-1}-a_nb_1)(a_{n-2}-a_nb_0)b_1)b_0^{n-2}$$
$$+ (-2(a_{n-2}-a_nb_0)a_{n-4} - (a_{n-2}-a_nb_0)a_{n-3}b_1 + C)b_0^{n-3} + P_{n-4}(b_0)$$

$$= (a_{n-1}^2 - 2a_na_{n-1}b_1 + a_n^2b_1^2)b_0^{n-1}$$
$$+ ((a_{n-2}^2 \underline{-2a_na_{n-2}b_0} + \underline{\underline{a_n^2b_0^2}}) + (-2a_{n-1}a_{n-3} + 2a_na_{n-3}b_1) + (-a_{n-1}a_{n-2}b_1 + a_na_{n-2}b_1^2 + \underline{a_na_{n-1}b_1b_0 - a_n^2b_1^2b_0}))b_0^{n-2}$$
$$+ ((-2a_{n-2}a_{n-4}\underline{+2a_na_{n-4}b_0}) + (-a_{n-2}a_{n-3}b_1 + \underline{a_na_{n-3}b_1b_0}) + C)b_0^{n-3} + P_{n-4}(b_0),$$

where C consists of all the terms which do not depend on b_0. Now we simplify the terms, collect the coefficients of b_0^n, b_0^{n-1} and b_0^{n-2} and put the remaining terms in a polynomial $P_{n-3}(b_0)$, which is of degree $\leq n-3$.

$$Res(p,q) = a_n^2b_0^n + (a_{n-1}^2 - \underline{2a_na_{n-1}b_1} + \underline{a_n^2b_1^2} - 2a_na_{n-2} + \underline{a_na_{n-1}b_1} - \underline{a_n^2b_1^2})b_0^{n-1}$$
$$+(a_{n-2}^2 - 2a_{n-1}a_{n-3} + \underline{2a_na_{n-3}b_1} - a_{n-1}a_{n-2}b_1 + a_na_{n-2}b_1^2 + 2a_na_{n-4} + \underline{a_na_{n-3}b_1})b_0^{n-2} + P_{n-3}(b_0)$$
$$= a_n^2b_0^n + (a_{n-1}^2 - a_na_{n-1}b_1 - 2a_na_{n-2})b_0^{n-1}$$
$$+ (a_{n-2}^2 - 2a_{n-1}a_{n-3} + 3a_na_{n-3}b_1 - a_{n-1}a_{n-2}b_1 + a_na_{n-2}b_1^2 + 2a_na_{n-4})b_0^{n-2} + P_{n-3}(b_0).$$

(iii) is a consequence of (ii). #

Note: The formulas (ii) and (iii) are valid for $n = 3$ if we define $a_{-1} := 0$.

1.2 Tschirnhaus' Method

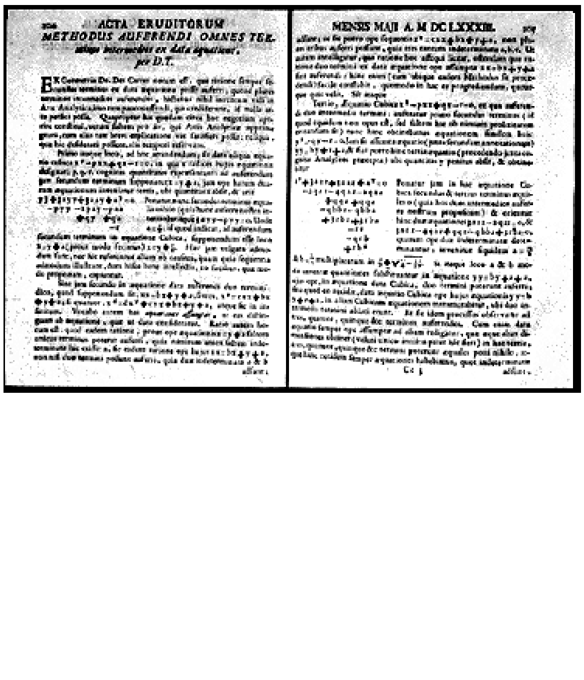
In his famous article *"Methodus auferendi omnes terminos intermedios ex data æquatione"* Tschirnhaus published a method how to remove the $(n-2)^{\text{th}}$ term in a polynomial $p(x) := x^n + a_{n-2}x^{n-2} + ... + a_0$ (where the $(n-1)^{\text{th}}$ term has already been canceled) by using a transformation.
Tschirnhaus described his method in the case $n = 3$ and discussed afterwards how to apply this method for $n = 4, 5, 6$.
Instead of regarding these special cases only we will describe his ideas for the general n-th order polynomial.

Tschirnhaus,
Ehrenfried Walter von,
** 10.04.1651*
in Kieslingswalde/Germany
† 11.10.1708
in Dresden/Germany

Tschirnhaus (also Tschirnhausen or Tzirnhausen), was born as son of the Kursächsischer Rat and Landesältester Christoph von Tschirnhausen and Elisabeth Eleonore Freiin Achyll von Stirling. After receiving private lessons at his parents house and attending the Gymnasium in Görlitz, he entered the university of Leiden in 1668 and studied there law, medicine, mathematics, science and Cartesian philosophy. 1672 he was volunteer in the Dutch army, when war broke out between the Netherlands and France.
Between 1674 and 1682 he was travelling through Europe, visiting Oxford, London, Lyon, Rome, Geneva etc. and getting in contact to Boyle, Newton, Leibniz and Huygens. Afterwards he published his research results (including ideas of constructing burning mirrors and his famous mathematical discovery of how to remove the terms x^{n-1} and x^{n-2} in an equation of degree n) in the Acta Eruditorum.
In 1682 he travelled to Paris and got there a member of the Académie Royal des Sciences.
In the same year he married Elisabeth Eleonore von Lest, who died in 1693. He had five children with her.
In winter 1693/94 he began his work of manufacturing porcellain, since 1696 supported by Kurfürst Friedrich August I, and in 1710, after his death, a factory in Meißen started producing his porcellain.
In 1688 he was accused of atheism by Christian Thomasius and his dispute with the priest of Kieslingswalde was continued till his death in 1694, when it was refused to him to have a church funeral.

Pages 206-207 of his famous mathematical article:

Methodus auferendi omnes terminos intermedios ex data æquatione,
in: Acta Eruditorum 1682, 204-207.


206 ACTA ERUDITORUM
METHODUS AUFERENDI OMNES TERminos intermedios ex data æquatione, per D.T.

MENSIS MAJI A. M DC LXXXII. 207

Suppose the polynomial $p(x) := x^n + a_{n-1}x^{n-1} + ... + a_0$ and a transformation of the form $y = x^m + b_{m-1}x^{m-1} + ... + b_0$. Substituding the variable y in the polynomial p leads to a new polynomial of the form $\tilde{p}(y) = y^n + c_{n-1}(b_{m-1}, ..., b_0)\ y^{n-1} + ... + c_0(b_{m-1}, ..., b_0)$. The aim is now to choose the m parameters $b_{m-1}, ..., b_0$ in such a way that the coefficients $c_{n-1}(b_{m-1}, ..., b_0), ..., c_{n-m}(b_{m-1}, ..., b_0)$ vanish.
Tschirnhaus believed (in the beginning) that it must be possible in that way to reduce $p(x) := x^n + a_{n-1}x^{n-1} + ... + a_0$ to the form $\tilde{p}(y) = y^n + c_0(b_{n-2}, ..., b_0)$ by using a transformation $y = x^{n-1} + b_{n-2}x^{n-2} + ... + b_0$ of degree $n-1$. After doing this the equation $\tilde{p}(y) = 0$ can be solved in y, and substituding y in $y = x^{n-1} + b_{n-2}x^{n-2} + ... + b_0$ leads to an equation of degree $n-1$ only. Solving this equation for x yields a solution of the n-th order equation $p(x) = 0$. So by induction it must be possible to solve the n-th order equation in a general way by radicals (in contradiction to Abel).
Tschirnhaus' way to find the 'right' parameters $b_{m-1}, ..., b_0$ was solving a suitable system of equations. We will go a little different and more formal way by using the resultant of the section before.
Regard the polynomials $p(x) := x^n + a_{n-1}x^{n-1} + ... + a_0$ and $q(x) = x^m + b_{m-1}x^{m-1} + ... + b_0 - y$. The resultant of both is a polynomial of the variable y only, i.e. $r(y) := Res(p, q)$, and as y appears in the corresponding determinant only in the last n columns, $r(y)$ is a polynomial of degree n. In this new polynomial we choose $b_{m-1}, ..., b_0$ such that the terms $n-1$ to $n-m$ vanish. So $q(x)$ is determined except the 'value' of y.
By Theorem 1.1.7 we know that $r(y) = 0$ is equivalent to a non-constant common divisor of p and q, what means that p and q have a common root (German: gemeinsame Nullstelle, Wurzel). So if we solve the equation $r(y) = 0$ in y and substitude this value of y in $q(x)$ (which is now determined completely) we know that both polynomials have a common root which can be found by solving $q(x) = 0$ (which is of lower degree than $p(x) = 0$) or, maybe much easier, by determing the GCD of p and q and determing its roots.
We will start with removing the $(n-1)^{\text{th}}$ term of $p(x) := x^n + a_{n-1}x^{n-1} + ... + a_0$.
Defining $q(x) = x + b_0 - y$ we know from 1.1.9 that

$$\begin{aligned} Res(p,q) &= (b_0 - y)^n + \textstyle\sum_{k=0}^{n-1} a_k(-1)^{n-k}(b_0 - y)^k \\ &= (b_0 - y)^n - a_{n-1}(b_0 - y)^{n-1} + \textstyle\sum_{k=0}^{n-2} a_k(-1)^{n-k}(b_0 - y)^k \\ &= (-1)^n y^n + (-1)^{n-1} n b_0 y^{n-1} + \textstyle\sum_{j=0}^{n-2} \binom{n}{j}(-1)^j y^j b_0^{n-j} && \text{binomial} \\ &\quad -a_{n-1}(-1)^{n-1}y^{n-1} - a_{n-1}\textstyle\sum_{j=0}^{n-2}\binom{n-1}{j}(-1)^j y^j b_0^{n-1-j} && \text{sum} \\ &\quad + \textstyle\sum_{k=0}^{n-2} a_k(-1)^{n-k}(b_0 - y)^k \\ &= (-1)^n y^n + (-1)^{n-1}(nb_0 - a_{n-1})y^{n-1} + \textstyle\sum ..., \end{aligned}$$

so the term $(-1)^{n-1}(nb_0 - a_{n-1})y^{n-1}$ vanishes if we choose $b_0 = \frac{a_{n-1}}{n}$.

1.2.1 Theorem By defining $y := x + \frac{a_{n-1}}{n}$ the polynomial $p(x) := x^n + a_{n-1}x^{n-1} + ... + a_0 \in \mathbb{F}[x]$ can be transformed to a polynomial $r(y)$ of degree n where the coefficient of the term y^{n-1} equals 0.

Now suppose we have transformed our polynomial in that way, i.e. we regard a polynomial $p(x)$ of the form $p(x) := x^n + a_{n-2}x^{n-2} + ... + a_0$ with no $(n-1)^{\text{th}}$ term and $a_{n-2} \neq 0$. To remove the $(n-2)^{\text{th}}$ term, too, we may assume a polynomial $q(x) := x^2 + b_1 x + b_0 - y$ and proceed in the same manner. By lemma 1.1.10 we know that

$$Res(p,q) = (b_0 - y)^n + A_{n-1}(b_0 - y)^{n-1} + A_{n-2}(b_0 - y)^{n-2} + P_{n-3}(y)$$

with $A_{n-1} = -2a_{n-2}$, $A_{n-2} = a_{n-2}^2 + 2a_{n-4} + 3a_{n-3}b_1 + a_{n-2}b_1^2$, since $a_{n-1} = 0$, and $P_{n-3}(y)$ a polynomial in y of degree $\leq n-3$.
Using again the binomial sum formula we obtain

$(b_0 - y)^n = (-1)^n y^n + (-1)^{n-1} n b_0 y^{n-1} + (-1)^{n-2} \frac{n(n-1)}{2} b_0^2 y^{n-2} \sum_{j=0}^{n-3} \binom{n}{j} (-1)^j y^j b_0^{n-j}$,
$(b_0 - y)^{n-1} = (-1)^{n-1} y^{n-1} + (-1)^{n-2}(n-1) b_0 y^{n-2} + \sum_{j=0}^{n-3} \binom{n-1}{j} (-1)^j y^j b_0^{n-1-j}$ and
$(b_0 - y)^{n-2} = (-1)^{n-2} y^{n-2} + \sum_{j=0}^{n-3} \binom{n-2}{j} (-1)^j y^j b_0^{n-2-j}$.

Substituding these expressions in the above formula and collecting all terms where y is of degree $\leq n-3$ in the polynomial $Q_{n-3}(y)$ we get

$$Res(p,q) = (-1)^n y^n + (-1)^{n-1}(nb_0 - 2a_{n-2}) y^{n-1}$$
$$+ (-1)^{n-2}\left(\tfrac{n(n-1)}{2} b_0^2 - (n-1)b_0 \cdot 2a_{n-2} + a_{n-2}^2 + 2a_{n-4} + 3a_{n-3}b_1 + a_{n-2}b_1^2\right) y^{n-2} + Q_{n-3}(y).$$

Since the $(n-1)^{\text{th}}$ and $(n-2)^{\text{th}}$ coefficient shall vanish we set both of them equal to zero and get a system of two equations. Solving the first yields b_0 and after substituding b_0 in the second equation we can solve it in terms of b_1:

Solve$[n\, b0 - 2\, a_{n-2} == 0,\ b0]$

$\{\{b0 \to \frac{2\, a_{-2+n}}{n}\}\}$

Collect$\left[\frac{n\,(n-1)}{2}\, b0^2 + (-2\, a_{n-2})\,(n-1)\, b0 + (a_{n-2}{}^2 + 2\, a_{n-4} + 3\, a_{n-3}\, b1 + a_{n-2}\, b1^2),\ b1\right]$

$2\, a_{-4+n} + 3\, b1\, a_{-3+n} + b1^2\, a_{-2+n} + a_{-2+n}^2 - \frac{2\,(-1+n)\, a_{-2+n}^2}{n}$

FullSimplify$\left[\textbf{Solve}\left[2\, a_{-4+n} + 3\, b1\, a_{-3+n} + b1^2\, a_{-2+n} + a_{-2+n}^2 - \frac{2\,(-1+n)\, a_{-2+n}^2}{n} == 0,\ b1\right]\right]$

$\left\{\left\{b1 \to \frac{-3\sqrt{n}\, a_{-3+n} - \sqrt{9\, n\, a_{-3+n}^2 - 8\, n\, a_{-4+n}\, a_{-2+n} + 4\,(-2+n)\, a_{-2+n}^3}}{2\sqrt{n}\, a_{-2+n}}\right\},\right.$

$\left.\left\{b1 \to \frac{-3\sqrt{n}\, a_{-3+n} + \sqrt{9\, n\, a_{-3+n}^2 - 8\, n\, a_{-4+n}\, a_{-2+n} + 4\,(-2+n)\, a_{-2+n}^3}}{2\sqrt{n}\, a_{-2+n}}\right\}\right\}$

1.2.2 Theorem By defining $y := x^2 + \left(-\frac{3}{2}\frac{a_{n-3}}{a_{n-2}} \pm \frac{1}{2}\sqrt{9\left(\frac{a_{n-3}}{a_{n-2}}\right)^2 - 8\frac{a_{n-4}}{a_{n-2}} + 4\frac{n-2}{n}a_{n-2}}\right)x + \frac{2a_{n-2}}{n}$ the polynomial $p(x) := x^n + a_{n-2}x^{n-2} + ... + a_0 \in \mathbb{F}[x]$ with $a_{n-2} \neq 0$ can be transformed in a polynomial $r(y)$ of degree n where the coefficients of the terms y^{n-1} and y^{n-2} equal 0.

As mentioned before, Tschirnhaus did not prove these general formulas above. He presented his ideas with the example of the cubic equation (see later) and discussed afterwards how to choose the above terms in the cases 4, 5, 6. He also claimed that using his ideas it would be possible to remove all the 'middle terms' of an polynomial equation which could afterwards be solved. Certainly he gave no prove for this claim since it is impossible to solve equations of degree higher than four by radicals only (→ Abel's Theorem), and another famous mathematician of these years, Gottfried Wilhelm Leibniz, didn't believe "that it will be succesful for equations of higher degree, except in special cases." (Kracht/Kreyszig [6]). By the way, it is interesting to note that Leibniz was in general very critical against Tschirnhaus, after their first contacts Leibniz stated: "Tschirnhaus had not really penetrated to any fundamental level of mathematical understanding. He remained trapped on the surface." (Kracht/Kreyszig [6]).
On the other hand, Tschirnhaus' method is indeed not succesful for removing more than two terms in general. We will see this during the discussion of Bring's ideas, that we have to solve equations of degree at least 6 in these cases.

1.2.3 Example Suppose the general third order equation $x^3 + px + q = 0$, where the quadratic term has already been removed. If we want to remove the linear term, too, we have to compare it with the equation $x^2 + b_1 x + b_0 - y = 0$.

Collect[Resultant[$q + p\,x + x^3$, $b_0 - y + b_1\,x + x^2$, x], y]

$$q^2 - y^3 + p^2 b_0 - 2\,p\,b_0^2 + b_0^3 + y^2\,(-2\,p + 3\,b_0) - p\,q\,b_1 + 3\,q\,b_0\,b_1 + p\,b_0\,b_1^2 - q\,b_1^3 + y\,(-p^2 + 4\,p\,b_0 - 3\,b_0^2 - 3\,q\,b_1 - p\,b_1^2)$$

Simplify[Solve[{$-2\,p + 3\,b_0 == 0$, $-p^2 + 4\,p\,b_0 - 3\,b_0^2 - 3\,q\,b_1 - p\,b_1^2 == 0$}, {$b_0$, b_1}]]

$$\left\{\left\{b_0 \to \frac{2\,p}{3},\ b_1 \to -\frac{9\,q + \sqrt{12\,p^3 + 81\,q^2}}{6\,p}\right\},\ \left\{b_0 \to \frac{2\,p}{3},\ b_1 \to \frac{-9\,q + \sqrt{12\,p^3 + 81\,q^2}}{6\,p}\right\}\right\}$$

It is easy to verify that these values for b_1 and b_0 correspond to the formula of Theorem 1.2.2 with $a_{-1} := 0$. Substituding the first pair in the resultant yields

Collect[Simplify[$q^2 - y^3 + p^2\,b_0 - 2\,p\,b_0^2 + b_0^3 + y^2\,(-2\,p + 3\,b_0) - p\,q\,b_1 + 3\,q\,b_0\,b_1 + p\,b_0\,b_1^2 - q\,b_1^3 + y\,(-p^2 + 4\,p\,b_0 - 3\,b_0^2 - 3\,q\,b_1 - p\,b_1^2)$], y]

$$\frac{8\,p^3}{27} + 4\,q^2 - \frac{2}{9}\,q\,\sqrt{12\,p^3 + 81\,q^2} - \frac{3\,q^3\,\left(-9\,q + \sqrt{12\,p^3 + 81\,q^2}\right)}{2\,p^3} - y^3$$

which is a cubic equation of the type $a - y^3 = 0$ and can therefore be solved by taking the cubic root.

$$\left\{\left\{y \to \frac{\left(\frac{(4\,p^3 + 27\,q^2)\,\left(4\,p^3 - 3\,q\,\left(-9\,q + \sqrt{12\,p^3 + 81\,q^2}\right)\right)}{p^3}\right)^{1/3}}{3\,2^{1/3}}\right\},$$

$$\{y \to -\frac{1}{3}\left(-\frac{1}{2}\right)^{1/3}\left(\frac{(4p^3+27q^2)\left(4p^3-3q\left(-9q+\sqrt{12p^3+81q^2}\right)\right)}{p^3}\right)^{1/3}\},$$

$$\{y \to \frac{(-1)^{2/3}\left(\frac{(4p^3+27q^2)\left(4p^3-3q\left(-9q+\sqrt{12p^3+81q^2}\right)\right)}{p^3}\right)^{1/3}}{3\,2^{1/3}}\}\}$$

By defining $A := \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}$ the first of the three solutions can be brought to the form $y = \frac{6A}{p}\sqrt[3]{A - \frac{q}{2}}$. After substituding y, b_0 and b_1 in $x^2 + b_1x + b_0 - y = 0$ we obtain the quadratic equation

$$x^2 + \frac{3}{p}(A - \frac{q}{2})x + \frac{2p}{3} - \frac{6A}{p}\sqrt[3]{A - \frac{q}{2}} = 0.$$

To find the common root of this and the original equation $x^3 + px + q = 0$ we can calculate the GCD of them and finally we get a result for x known as Cardanos formula. For a more detailed discussion of these last steps see Tignol [8].

1.2.4 Example Let us finally look at the quartic equation $x^4 + px^2 + qx + r = 0$ without a cubic term. Again we will compare it to the equation $x^2 + b_1x + b_0 - y = 0$.

Collect[Resultant[x⁴ + p x² + q x + r, x² + b1 x + b0 - y, x], y]

$$b0^4 - 2\,b0^3\,p + b0^2\,b1^2\,p + b0^2\,p^2 + 3\,b0^2\,b1\,q - b0\,b1^3\,q - b0\,b1\,p\,q + b0\,q^2 + 2\,b0^2\,r - 4\,b0\,b1^2\,r + b1^4\,r - 2\,b0\,p\,r + b1^2\,p\,r - b1\,q\,r + r^2 + (-4\,b0^3 + 6\,b0^2\,p - 2\,b0\,b1^2\,p - 2\,b0\,p^2 - 6\,b0\,b1\,q + b1^3\,q + b1\,p\,q - q^2 - 4\,b0\,r + 4\,b1^2\,r + 2\,p\,r)\,y + (6\,b0^2 - 6\,b0\,p + b1^2\,p + p^2 + 3\,b1\,q + 2\,r)\,y^2 + (-4\,b0 + 2\,p)\,y^3 + y^4$$

Solve[-4 b0 + 2 p == 0, b0]

$$\{\{b0 \to \frac{p}{2}\}\}$$

But instead of removing the quadratic term we will cancel out the coefficient of y, i.e., we have to find the roots in b_1 of the following cubic polynomial:

Collect[-4 b0³ + 6 b0² p - 2 b0 b1² p - 2 b0 p² -
6 b0 b1 q + b1³ q + b1 p q - q² - 4 b0 r + 4 b1² r + 2 p r, b1]

$$b1^3\,q - 2\,b1\,p\,q - q^2 + b1^2\,(-p^2 + 4\,r)$$

We won't replace b_1 by one of its value, because it would take really much place, so we will only substitude b_0 in the resultant:

```
Collect[b0^4 - 2 b0^3 p + b0^2 b1^2 p + b0^2 p^2 + 3 b0^2 b1 q -
  b0 b1^3 q - b0 b1 p q + b0 q^2 + 2 b0^2 r - 4 b0 b1^2 r + b1^4 r - 2 b0 p r +
  b1^2 p r - b1 q r + r^2 + (6 b0^2 - 6 b0 p + b1^2 p + p^2 + 3 b1 q + 2 r) y^2 + y^4, y]
```

$$\frac{b1^2\,p^3}{4}+\frac{p^4}{16}-\frac{1}{2}\,b1^3\,p\,q+\frac{1}{4}\,b1\,p^2\,q+\frac{p\,q^2}{2}+b1^4\,r-b1^2\,p\,r-\frac{p^2\,r}{2}-b1\,q\,r+r^2+\left(b1^2\,p-\frac{p^2}{2}+3\,b1\,q+2\,r\right)y^2+y^4$$

This is a quadratic equation in y^2 and can therefore be solved easily:

```
FullSimplify[Solve[b1^2 p^3/4 + p^4/16 - 1/2 b1^3 p q + 1/4 b1 p^2 q + p q^2/2 + b1^4 r -
    b1^2 p r - p^2 r/2 - b1 q r + r^2 + (b1^2 p - p^2/2 + 3 b1 q + 2 r) y2 + y2^2 == 0, y2]]
```

$$\Big\{\Big\{y2\to\frac{1}{4}\Big(-2\,b1^2\,p+p^2-6\,b1\,q-2\Big(2\,r+\sqrt{8\,b1^3\,p\,q-2\,p\,q^2+b1^4\,(p^2-4\,r)-4\,b1\,q\,(p^2-4\,r)+b1^2\,(-2\,p^3+9\,q^2+8\,p\,r)}$$

$$\Big\{y2\to\frac{1}{4}\Big(-2\,b1^2\,p+p^2-6\,b1\,q-4\,r+2\sqrt{8\,b1^3\,p\,q-2\,p\,q^2+b1^4\,(p^2-4\,r)-4\,b1\,q\,(p^2-4\,r)+b1^2\,(-2\,p^3+9\,q^2+8\,p\,r)}\Big)\Big\}\Big\}$$

Now we "only" have to take the squareroot of one of these values and substitude b_1 to get a solution y of the resultant. Afterwards we substitude this value of y and the values of b_1 and b_0 in $x^2 + b_1 x + b_0 - y = 0$ and look again for the GCD with the original equation $x^4 + px^2 + qx + r = 0$.

After illustrading Tschirnhaus' ideas with this two examples we will look at the developments of the Tschirnhaus transformations by some other mathematicians.

Chapter 2

Developments based on Tschirnhaus' Idea

2.1 Bezout's Method

About 1765 Bezout developped a new method of solving equations. He used a similar idea like Tschirnhaus to express the roots of a polynomial as a polynomial in a root of unity.
The idea is quite simple: Instead of comparing a given equation $x^n + c_{n-1}x^{n-1} + ... + c_0 = 0$ and a polynomial of the type $x^m + b_{m-1}x^{m-1} + ... + b_0 = y$ like Tschirnhaus, Bezout looked at the equations $x = a_{n-1}y^{n-1} + ... + a_0$ and $y^n - 1 = 0$ and tried to eliminate y between them. The result should be a polynomial of degree n in x, and this polynomial will be compared to our given polynomial.
Let's describe his idea more detailed: If we define

$$r_n(x) = Res(a_{n-1}y^{n-1} + ... + a_1y + a_0 - x, y^n - 1),$$

then $r_n(x)$ is a polynomial of degree n. By Theorem 1.1.7 we know that $r_n(x) = 0$ is equivalent to the fact that the polynomials $a_{n-1}y^{n-1} + ... + a_1y + a_0 - x$ and $y^n - 1$ have a common root.

Bézout,
Etienne,
** 31.03.1730*
in Nemours/France
† 27.09.1783
in Basses-Loges/France

Bézout didn't follow the family tradition of being magistrate in the town of Nemours. After reading Euler's works he decided to become a mathematician.
1757 he published "Quantités différentielles" and 1758 "Rectification des courbes". In the same year he was appointed an adjoint in mechanics of the Académie des Sciences and became royal censor.
Bézout got very famous for his textbooks specially designed for teaching students. Some of them were translated into English and used at the Harvard University in North America.
But Bézout is also famed for his work on algebra, in particular for his theories on equations.
In his work "Sur plusieurs classes d'equations de tous les degrés qui admettent une solution algébrique" he described how to write a single equation as two equations in two unknowns where one of these equations consists of only two terms, the term of degree n and a constant term.

But by the second polynomial we know that this common root must be an n-th root of unity ξ, so substituding $y = \xi$ in the first polynomial yields $x = a_{n-1}\xi^{n-1} + ... + a_1\xi + a_0$, so the root x of $r_n(x)$ has a polynomial expression in a root of unity.
The idea will get more clear if we look at some examples. But first we calculate some of the polynomials $r_n(x)$, where $P := x - a_0$:

Collect[Resultant[-P + a1 y, y^2 - 1, y], P]

$-a1^2 + P^2$

Collect[Resultant[-P + a1 y + a2 y^2, y^3 - 1, y], P]

$a1^3 + a2^3 + 3\,a1\,a2\,P - P^3$

Collect[Resultant[-P + a1 y + a2 y^2 + a3 y^3, y^4 - 1, y], P]

$-a1^4 + a2^4 - 4\,a1\,a2^2\,a3 + 2\,a1^2\,a3^2 - a3^4 + (-4\,a1^2\,a2 - 4\,a2\,a3^2)\,P + (-2\,a2^2 - 4\,a1\,a3)\,P^2 + P^4$

Collect[Resultant[-P + a1 y + a2 y^2 + a3 y^3 + a4 y^4, y^5 - 1, y], P]

$a1^5 + a2^5 - 5\,a1\,a2^3\,a3 + 5\,a1^2\,a2\,a3^2 + a3^5 + 5\,a1^2\,a2^2\,a4 -$
$5\,a1^3\,a3\,a4 - 5\,a2\,a3^3\,a4 + 5\,a2^2\,a3\,a4^2 + 5\,a1\,a3^2\,a4^2 - 5\,a1\,a2\,a4^3 + a4^5 +$
$(5\,a1^3\,a2 - 5\,a2^2\,a3^2 + 5\,a1\,a3^3 + 5\,a2^3\,a4 + 5\,a1\,a2\,a3\,a4 - 5\,a1^2\,a4^2 + 5\,a3\,a4^3)\,P +$
$(5\,a1\,a2^2 + 5\,a1^2\,a3 + 5\,a3^2\,a4 + 5\,a2\,a4^2)\,P^2 + (5\,a2\,a3 + 5\,a1\,a4)\,P^3 - P^5$

Collect[Resultant[-P + a1 y + a2 y^2 + a3 y^3 + a4 y^4 + a5 y^5, y^6 - 1, y], P]

$-a1^6 + a2^6 - 6\,a1\,a2^4\,a3 + 9\,a1^2\,a2^2\,a3^2 - 2\,a1^3\,a3^3 - a3^6 + 6\,a1^2\,a2^3\,a4 - 12\,a1^3\,a2\,a3\,a4 +$
$6\,a2\,a3^4\,a4 + 3\,a1^4\,a4^2 - 9\,a2^2\,a3^2\,a4^2 - 6\,a1\,a3^3\,a4^2 + 2\,a2^3\,a4^3 + 12\,a1\,a2\,a3\,a4^3 - 3\,a1^2\,a4^4 +$
$a4^6 - 6\,a1^3\,a2^2\,a5 + 6\,a1^4\,a3\,a5 - 6\,a2^2\,a3^3\,a5 + 6\,a1\,a3^4\,a5 + 12\,a2^3\,a3\,a4\,a5 - 18\,a1\,a2^2\,a4^2\,a5 -$
$6\,a3\,a4^4\,a5 - 3\,a2^4\,a5^2 - 9\,a1^2\,a3^2\,a5^2 + 18\,a1^2\,a2\,a4\,a5^2 + 9\,a3^2\,a4^2\,a5^2 + 6\,a2\,a4^3\,a5^2 -$
$2\,a1^3\,a5^3 - 2\,a3^3\,a5^3 - 12\,a2\,a3\,a4\,a5^3 - 6\,a1\,a4^2\,a5^3 + 3\,a2^2\,a5^4 + 6\,a1\,a3\,a5^4 - a5^6 +$
$(-6\,a1^4\,a2 - 6\,a2^3\,a3^2 + 12\,a1\,a2\,a3^3 + 6\,a2^4\,a4 - 18\,a1^2\,a3^2\,a4 - 6\,a3^2\,a4^3 + 6\,a2\,a4^4 - 12\,a1\,a2^3$
$a5 + 12\,a1^3\,a4\,a5 + 12\,a3^3\,a4\,a5 - 12\,a1\,a4^3\,a5 - 18\,a2\,a3^2\,a5^2 + 12\,a1\,a2\,a5^3 - 6\,a4\,a5^4)\,P +$
$(-9\,a1^2\,a2^2 - 6\,a1^3\,a3 + 3\,a3^4 + 9\,a2^2\,a4^2 - 18\,a1\,a3\,a4^2 - 18\,a2^2\,a3\,a5 + 9\,a1^2\,a5^2 -$
$9\,a4^2\,a5^2 - 6\,a3\,a5^3)\,P^2 + (-2\,a2^3 - 12\,a1\,a2\,a3 - 6\,a1^2\,a4 - 2\,a4^3 - 12\,a3\,a4\,a5 - 6\,a2\,a5^2)$
$P^3 + (-3\,a3^2 - 6\,a2\,a4 - 6\,a1\,a5)\,P^4 + P^6$

2.1.1 Example Let's look at the third order polynomial $p(x) = x^3 + px + q$ without quadratic term. If we compare it with

$$r_3(x) = -(x - a_0)^3 + 3a_1a_2(x - a_0) + a_1^3 + a_2^3$$

(remember that $P := x - a_0$) we can choose $a_0 = 0$ (or $P = x$) and have to solve $-3a_1a_2 = p$ and $-a_1^3 - a_2^3 = q$:

```
FullSimplify[Solve[{-3 a1 a2 == p, -a1^3 - a2^3 == q}, {a1, a2}]]
```

$$\left\{\left\{a1 \to -\frac{p}{3\left(-\frac{q}{2}-\frac{1}{54}\sqrt{108\,p^3+729\,q^2}\right)^{1/3}},\ a2 \to \left(-\frac{q}{2}-\frac{1}{54}\sqrt{108\,p^3+729\,q^2}\right)^{1/3}\right\},\right.$$

As result we get 6 pairs of solutions (only the first of them is shown above) and we choose the first pair of the solutions. Substituding them in $a_2y^2 + a_1y - x$ yields

```
Collect[-x + a1 y + a2 y^2, y]
```

$$-x - \frac{p\,y}{3\left(-\frac{q}{2}-\frac{1}{54}\sqrt{108\,p^3+729\,q^2}\right)^{1/3}} + \left(-\frac{q}{2}-\frac{1}{54}\sqrt{108\,p^3+729\,q^2}\right)^{1/3} y^2$$

$$\mathbf{x = -\frac{p\,y}{3\left(-\frac{q}{2}-\frac{1}{54}\sqrt{108\,p^3+729\,q^2}\right)^{1/3}} + \left(-\frac{q}{2}-\frac{1}{54}\sqrt{108\,p^3+729\,q^2}\right)^{1/3} y^2}$$

As y is a third root of unity (i.e. y equals 1, $e^{\frac{\imath\pi}{3}}$ or $e^{\frac{2\imath\pi}{3}}$), we found an expression of one root of $x^3 + px + q = 0$ as polynomial in the third roots of unity. And if we choose for example $y = 1$ then we get

```
FullSimplify[x]
```

$$-\frac{p}{3\left(-\frac{q}{2}-\frac{1}{54}\sqrt{108\,p^3+729\,q^2}\right)^{1/3}} + \left(-\frac{q}{2}-\frac{1}{54}\sqrt{108\,p^3+729\,q^2}\right)^{1/3}$$

To check that this is indeed a solution of the given equation $x^3 + px + q = 0$ we substitude this result of x in the equation:

```
FullSimplify[x^3 + p x + q]
```

0

2.1.2 Example As a second example we look at the fourth order equation $x^4 + px^2 + qx + r = 0$ without cubic term. If we compare this with

$$r_4(x) = (x+a_0)^4 + (-2a_2^2 - 4a_1a_3)(x-a_0)^2 + (-4a_1^2a_2 - 4a_2a_3^2)(x-a_0) - a_1^4 + a_2^4 - a_3^4 - 4a_1a_2^2a_3 + 2a_1^2a_3^2$$

we notice that we can again choose $a_0 = 0$ and solve then in a second step the equations $-2a_2^2 - 4a_1a_3 = p$ and $-4a_1^2a_2 - 4a_2a_3^2 = q$ for a_1 and a_3:

```
FullSimplify[Solve[{-2 a2^2 - 4 a1 a3 == p, -4 a1^2 a2 - 4 a2 a3^2 == q}, {a1, a3}]]
```

$$\left\{\left\{a1 \to -\frac{2\,a2^2+p}{\sqrt{2}\sqrt{-\frac{q+\sqrt{-4\,a2^2\,(2\,a2^2+p)^2+q^2}}{a2}}},\ a3 \to \frac{\sqrt{-\frac{q+\sqrt{-4\,a2^2\,(2\,a2^2+p)^2+q^2}}{a2}}}{2\sqrt{2}}\right\},\right.$$

The result are four pairs of solution (again above is shown only the first of these pairs) and we choose the first for substituding it in the remaining equation $-a_1^4 + a_2^4 - a_3^4 - 4a_1a_2^2a_3 + 2a_1^2a_3^2 = r$:

```
FullSimplify[-a1^4 + a2^4 - 4 a1 a2^2 a3 + 2 a1^2 a3^2 - a3^4 == r]
```

$$\frac{1}{4}\,(4\,\mathrm{a2}^2 + \mathrm{p})^2 == \frac{\mathrm{q}^2}{16\,\mathrm{a2}^2} + \mathrm{r}$$

This equation can be brought to the form

$$a_2^6 + \frac{p}{2}a_2^4 + \left(\frac{p}{16} - \frac{r}{4}\right)a_2^2 - \frac{q^2}{64} = 0,$$

which is cubic in a_2^2 and can therefore be solved by using Cardano's formula first and the squareroot afterwards.

Note that this method yields no results (in general) if the degree n is larger than 4, as one can easily imagine by looking at $r_5(x)$ or $r_6(x)$ above, because in these cases we have to solve equations of degree larger than 4.
But nevertheless this is an interesting method, because it leads to an description of the roots of an n-th order polynomial by the n-th roots of unity!

2.2 Bring's Transformation of the Quintic

2.2.1 Example Suppose the quintic equation $x^5+a_2x^2+a_1x+a_0$ without cubic and fourth order term (the general fifth order polynomial can always be brought to this form by using Tschirnhaus transformations). We try now to remove the quadratic term, too, by using Tschirnhaus' idea, i.e. we compare it with the polynomial $x^3+b_2x^2+b_1x+b_0$ by calculating the resultant (only the fourth order, cubic and quadratic term are given completely below):

$$\mathbf{Collect[Simplify[Resultant[x^5 + a_2\,x^2 + a_1\,x + a_0,\ \ x^3 + b_2\,x^2 + b_1\,x + b_0 - y,\ x]],\ y]}$$

$$-y^5 - a_0^3 + a_1^3\,b_0 - 3\,a_0\,a_1\,a_2\,b_0 + 3\,a_0\,a_1\,b_0^2 - a_2^3\,b_0^2 + 3\,a_2^2\,b_0^3 - 3\,a_2\,b_0^4 + b_0^5 + y^4\,(-3\,a_2 + 5\,b_0) -$$

$$y^3\,(-3\,a_2^2 + 12\,a_2\,b_0 - 10\,b_0^2 - 4\,a_1\,b_1 - 5\,a_0\,b_2 - 3\,a_2\,b_1\,b_2 - 2\,a_1\,b_2^2) + y^2\,(3\,a_0\,a_1 - a_2^3 + 9\,a_2^2\,b_0 - 18\,a_2\,b_0^2 + 10\,b_0^3 - 5\,a_1\,a_2\,b_1 + 12\,a_1\,b_0\,b_1 - 5\,a_0\,b_1^2 - a_2\,b_1^3 + 4\,a_1^2\,b_2 - 7\,a_0\,a_2\,b_2 + 15\,a_0\,b_0\,b_2 - 3\,a_2^2\,b_1\,b_2 + 9\,a_2\,b_0\,b_1\,b_2 - 4\,a_1\,b_1^2\,b_2 + a_1\,a_2\,b_2^2 + 6\,a_1\,b_0\,b_2^2 - 5\,a_0\,b_1\,b_2^2 + a_2^2\,b_2^3) +$$

To remove the fourth order term we have to solve the following equation:

$$\mathbf{Solve[-3\,a_2 + 5\,b_0 == 0,\ b_0]}$$

$$\left\{\left\{b_0 \to \frac{3\,a_2}{5}\right\}\right\}$$

Now we substitude this value for b_0 in the coefficient of the third order term and determine afterwards b_1.

$$\mathbf{Simplify[-3\,a_2^2 + 12\,a_2\,b_0 - 10\,b_0^2 - 4\,a_1\,b_1 - 5\,a_0\,b_2 - 3\,a_2\,b_1\,b_2 - 2\,a_1\,b_2^2]}$$

$$\frac{3\,a_2^2}{5} - 5\,a_0\,b_2 - 3\,a_2\,b_1\,b_2 - 2\,a_1\,(2\,b_1 + b_2^2)$$

$$\mathbf{Solve\left[\frac{3\,a_2^2}{5} - 5\,a_0\,b_2 - 3\,a_2\,b_1\,b_2 - 2\,a_1\,(2\,b_1 + b_2^2) == 0,\ b_1\right]}$$

$$\left\{\left\{b_1 \to -\frac{-3\,a_2^2 + 25\,a_0\,b_2 + 10\,a_1\,b_2^2}{5\,(4\,a_1 + 3\,a_2\,b_2)}\right\}\right\}$$

Finally we substitude b_1 in the coefficient of the quadratic term which yields to a rational expression. If we now want to know the roots of this formula in b_2 we can multiply it with its main denominator and determine afterwards the roots of the resulting polynomial in b_2.

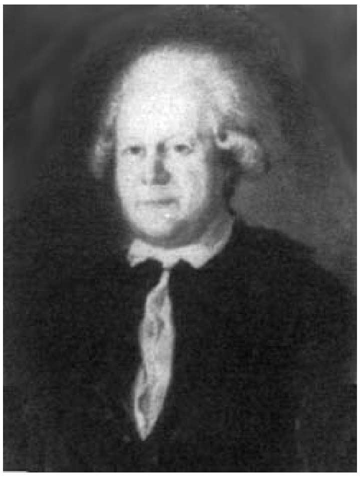
$$\mathbf{FullSimplify[3\,a_0\,a_1 - a_2^3 + 9\,a_2^2\,b_0 - 18\,a_2\,b_0^2 + 10\,b_0^3 - 5\,a_1\,a_2\,b_1 + 12\,a_1\,b_0\,b_1 - 5\,a_0\,b_1^2 - a_2\,b_1^3 + 4\,a_1^2\,b_2 - 7\,a_0\,a_2\,b_2 + 15\,a_0\,b_0\,b_2 - 3\,a_2^2\,b_1\,b_2 + 9\,a_2\,b_0\,b_1\,b_2 - 4\,a_1\,b_1^2\,b_2 + a_1\,a_2\,b_2^2 + 6\,a_1\,b_0\,b_2^2 - 5\,a_0\,b_1\,b_2^2 + a_2^2\,b_2^3]}$$

$$3 a_0 a_1 + \frac{2 a_2^3}{25} + 4 a_1^2 b_2 + 2 a_0 a_2 b_2 + \frac{23}{5} a_1 a_2 b_2^2 + a_2^2 b_2^3 +$$
$$\frac{12 a_2^2 b_2 (3 a_2^2 - 5 b_2 (5 a_0 + 2 a_1 b_2))}{25 (4 a_1 + 3 a_2 b_2)} - \frac{a_0 (3 a_2^2 - 5 b_2 (5 a_0 + 2 a_1 b_2))^2}{5 (4 a_1 + 3 a_2 b_2)^2} -$$
$$\frac{4 a_1 b_2 (3 a_2^2 - 5 b_2 (5 a_0 + 2 a_1 b_2))^2}{25 (4 a_1 + 3 a_2 b_2)^2} - \frac{11 a_1 a_2 (-3 a_2^2 + 5 b_2 (5 a_0 + 2 a_1 b_2))}{25 (4 a_1 + 3 a_2 b_2)} +$$
$$\frac{a_0 b_2^2 (-3 a_2^2 + 5 b_2 (5 a_0 + 2 a_1 b_2))}{4 a_1 + 3 a_2 b_2} + \frac{a_2 (-3 a_2^2 + 5 b_2 (5 a_0 + 2 a_1 b_2))^3}{125 (4 a_1 + 3 a_2 b_2)^3}$$

$$192 a_0 a_1^4 + \frac{656}{25} a_1^3 a_2^3 - \frac{36}{5} a_0 a_1 a_2^4 - \frac{27 a_2^7}{125} +$$
$$\left(256 a_1^5 + 384 a_0 a_1^3 a_2 + 120 a_0^2 a_1 a_2^2 + \frac{1512}{25} a_1^2 a_2^4\right) b_2 +$$
$$\left(-500 a_0^3 a_1 + 800 a_1^4 a_2 + 252 a_0 a_1^2 a_2^2 + 45 a_0^2 a_2^3 + \frac{1323}{25} a_1 a_2^5\right) b_2^2 +$$
$$\left(-400 a_0^2 a_1^2 - 250 a_0^3 a_2 + \frac{5072}{5} a_1^3 a_2^2 - 90 a_0 a_1 a_2^3 + \frac{378 a_2^6}{25}\right) b_2^3 +$$
$$\left(-240 a_0 a_1^3 + 150 a_0^2 a_1 a_2 + \frac{3078}{5} a_1^2 a_2^3 - 81 a_0 a_2^4\right) b_2^4 +$$
$$(-64 a_1^4 + 225 a_0^2 a_2^2 + 189 a_1 a_2^4) b_2^5 + (-40 a_1^3 a_2 + 90 a_0 a_1 a_2^2 + 27 a_2^5) b_2^6$$

But the result is a sixth order polynomial which is not solvable by radicals! So it is impossible to remove the fourth order, cubic AND quadratic term of a quintic polynomial by using Tschirnhaus transformations!

About hundred years after Tschirnhaus, the Swedish mathematician Erland Samuel Bring discovered a method how to remove also the quadratic term of a fifth order polynomial which was brought before to the form z^5+pz^2+qz+r by using Tschirnhaus transformations. The way he does that is described in Cantor [2] and we follow this description by using again mathematica 4.0.



Bring,
Erland Samuel,
** 19.08.1736*
in Ausas/Sweden
† 20.05.1798
in Lund/Sweden

Bring studied at Lund from 1750 to 1757 and began afterwards teaching there history. 1762 he became a reader and 1779 a professor.
His mathematical work contains algebra, geometry and analysis, and his most famous work, "Meletemata quaedam mathematica circa transformationem æquationum algebraicum" (1786), describes his contribution to the algebraic solution of equations. He found a method of transforming the general quintic equation to the form $x^5+px+q=0$.

First we start again by calculating the resultant of $z^5 + pz^2 + qz + r$ and $z^4 + dz^3 + cz^2 + bz + a + y$, but since the result fills one complete page only the quadratic, cubic and fourth order term are given below.

```
Collect[Simplify[Resultant[z^5 + p z^2 + q z + r, z^4 + d z^3 + c z^2 + b z + a + y, z]], y]
```

$$\begin{aligned}&(10 a^3 - b^3 p + 9 a b c p - 18 a^2 d p + 3 b^2 p^2 - 9 a c p^2 + c^3 p^2 - 3 b c d p^2 + 9 a d^2 p^2 -\\
&\quad 3 b p^3 + 3 c d p^3 - d^3 p^3 + p^4 - 24 a^2 q - 4 b^2 c q + 6 a c^2 q + 12 a b d q + 2 b c p q +\\
&\quad 15 a d p q + c^2 d p q - 5 b d^2 p q + 2 c p^2 q - d^2 p^2 q + 18 a q^2 - 4 c^2 q^2 - 8 b d q^2 +\\
&\quad 4 c d^2 q^2 - d p q^2 - 4 q^3 + 15 a b r - 5 b c^2 r - 5 b^2 d r + 15 a c d r - 12 a p r + 8 c^2 p r +\\
&\quad b d p r - 7 c d^2 p r + d p^2 r - 11 b q r - 2 c d q r + 3 d^3 q r + 8 p q r + 5 c r^2 + 5 d^2 r^2)\, y^2 +\\
&(10 a^2 + 3 b c p - 12 a d p - 3 c p^2 + 3 d^2 p^2 - 16 a q + 2 c^2 q + 4 b d q + 5 d p q +\\
&\quad 6 q^2 + 5 b r + 5 c d r - 4 p r)\, y^3 + (5 a - 3 d p - 4 q)\, y^4 + y^5\end{aligned}$$

We start again with the fourth order term and try to remove it.

```
Solve[5 a - 3 d p - 4 q == 0, a]
```

$$\left\{\left\{a \to \frac{1}{5}\,(3 d p + 4 q)\right\}\right\}$$

Afterwards we substitude the resulting value of a in the cubic and quadratic term.

$$\begin{aligned}&\Big(-b^3 p + 3 b^2 p^2 + c^3 p^2 + \frac{12}{5} b c d p^2 - 3 b p^3 - \frac{12}{5} c d p^3 + \frac{2 d^3 p^3}{25} + p^4 - 4 b^2 c q +\\
&\quad \frac{46}{5} b c p q + \frac{23}{5} c^2 d p q + \frac{11}{5} b d^2 p q - \frac{26}{5} c p^2 q - \frac{52}{25} d^2 p^2 q + \frac{4 c^2 q^2}{5} +\\
&\quad \frac{8}{5} b d q^2 + 4 c d^2 q^2 - \frac{31}{25} d p q^2 + \frac{4 q^3}{25} - 5 b c^2 r - 5 b^2 d r + 8 c^2 p r + 10 b d p r +\\
&\quad 2 c d^2 p r - \frac{31}{5} d p^2 r + b q r + 10 c d q r + 3 d^3 q r - \frac{8 p q r}{5} + 5 c r^2 + 5 d^2 r^2\Big)\, y^2 +\\
&\Big(3 b c p - 3 c p^2 - \frac{3 d^2 p^2}{5} + 2 c^2 q + 4 b d q - \frac{23 d p q}{5} - \frac{2 q^2}{5} + 5 b r + 5 c d r - 4 p r\Big)\, y^3 + y^5\end{aligned}$$

Now comes the clever idea of Bring: Bring assumed the parameters b and c to be of the form $b = \alpha d + \xi$ and $c = d + \gamma$, i.e. b and c are linear expressions of d. Substituding both in the cubic coefficient gives a quadratic polynomial in d which has to equal zero. Therefore the coefficients of d^0, d^1 and d^2 are set to be zero and we pick out the first result for α, ξ and γ.

$$\texttt{Collect}\Big[\texttt{Simplify}\Big[3 b c p - 3 c p^2 - \frac{3 d^2 p^2}{5} + 2 c^2 q + 4 b d q - \frac{23 d p q}{5} - \frac{2 q^2}{5} + 5 b r + 5 c d r - 4 p r\Big]$$

$$\begin{aligned}&-\frac{2 q^2}{5} - 4 p r + d^2 \left(-\frac{3 p^2}{5} + 2 q + 5 r + 3 p \alpha + 4 q \alpha\right) - 3 p^2 \gamma + 2 q \gamma^2 +\\
&\quad 5 r \xi + 3 p \gamma \xi + d \left(-3 p^2 - \frac{23 p q}{5} + 5 r \alpha + 4 q \gamma + 5 r \gamma + 3 p \alpha \gamma + 3 p \xi + 4 q \xi\right)\end{aligned}$$

```
Simplify[Solve[{-3 p^2/5 + 2 q + 5 r + 3 p α + 4 q α == 0,
  -3 p^2 - 23 p q/5 + 5 r α + 4 q γ + 5 r γ + 3 p α γ + 3 p ξ + 4 q ξ == 0,
  -2 q^2/5 - 4 p r - 3 p^2 γ + 2 q γ^2 + 5 r ξ + 3 p γ ξ == 0}, {α, ξ, γ}]]
```

$$\Big\{\alpha \to \frac{3p^2-10q-25r}{15p+20q},$$

$$\xi \to \Big(810p^6+1161p^5q-60p^2q^2(92q-235r)-9p^3\Big(30q^2-1200qr-125r^2+\sqrt{5}\sqrt{-27p^4q^2+256q^5+108p^5r-1600pq^3r+2250p^2qr^2+3125r^4}\Big)-10pq\Big(736q^3+375r^2+3\sqrt{5}\sqrt{-27p^4q^2+256q^5+108p^5r-1600pq^3r+2250p^2qr^2+3125r^4}\Big)+20q\Big(200q^2r+500qr^2+625r^3-4\sqrt{5}\,q\sqrt{-27p^4q^2+256q^5+108p^5r-1600pq^3r+2250p^2qr^2+3125r^4}-5\sqrt{5}\,r\sqrt{-27p^4q^2+256q^5+108p^5r-1600pq^3r+2250p^2qr^2+3125r^4}\Big)\Big)\Big/$$

$$(10(3p+4q)(27p^4-160q^3+300pqr)),\ \gamma \to$$

$$\Big(27p^3q+36p^2q^2-90p^3r-400q^2r+375pr^2-500qr^2+\sqrt{5}\,(3p+4q)\sqrt{-27p^4q^2+256q^5+108p^5r-1600pq^3r+2250p^2qr^2+3125r^4}\Big)\Big/$$

$$(2(27p^4-160q^3+300pqr))\Big\}$$

Now we can express b and c in d and substitude them in the quadratic coefficient. The result is a cubic polynomial in d which is 7(!) pages long. Representatively for the whole result the cubic term is shown below:

```
Simplify[b = α d + ξ]
```

$$\frac{d\,(3p^2-10q-25r)}{15p+20q}+$$

$$\Big(810p^6+1161p^5q-60p^2q^2(92q-235r)-9p^3\Big(30q^2-1200qr-125r^2+\sqrt{5}\sqrt{-27p^4q^2+256q^5+108p^5r-1600pq^3r+2250p^2qr^2+3125r^4}\Big)-10pq$$

$$\Big(736q^3+375r^2+3\sqrt{5}\sqrt{-27p^4q^2+256q^5+108p^5r-1600pq^3r+2250p^2qr^2+3125$$

$$20q\Big(200q^2r+500qr^2+625r^3-4\sqrt{5}\,q\sqrt{-27p^4q^2+256q^5+108p^5r-1600pq^3r+2250p^2qr^2+3125r^4}-5\sqrt{5}\,r\sqrt{-27p^4q^2+256q^5+108p^5r-1600pq^3r+2250p^2qr^2+3125r^4}\Big)\Big)\Big/$$

$$(10(3p+4q)(27p^4-160q^3+300pqr))$$

```
Simplify[c = d + γ]
```

$$d+\Big(27p^3q+36p^2q^2-90p^3r-400q^2r+375pr^2-500qr^2+\sqrt{5}\,(3p+4q)\sqrt{-27p^4q^2+256q^5+108p^5r-1600pq^3r+2250p^2qr^2+3125r^4}\Big)\Big/$$

$$(2(27p^4-160q^3+300pqr))$$

```
Collect[Simplify[-b^3 p + 3 b^2 p^2 + c^3 p^2 + 12/5 b c d p^2 - 3 b p^3 - 12/5 c d p^3 + (2 d^3 p^3)/25 + p^4 -
   4 b^2 c q + 46/5 b c p q + 23/5 c^2 d p q + 11/5 b d^2 p q - 26/5 c p^2 q - 52/25 d^2 p^2 q + (4 c^2 q^2)/5 +
   8/5 b d q^2 + 4 c d^2 q^2 - 31/25 d p q^2 + (4 q^3)/25 - 5 b c^2 r - 5 b^2 d r + 8 c^2 p r + 10 b d p r +
   2 c d^2 p r - 31/5 d p^2 r + b q r + 10 c d q r + 3 d^3 q r - (8 p q r)/5 + 5 c r^2 + 5 d^2 r^2], d]
```

$$d^3\left(p^2+\frac{2p^3}{25}+\frac{23pq}{5}+4q^2+\frac{12p^2(3p^2-10q-25r)}{5(15p+20q)}+\frac{11pq(3p^2-10q-25r)}{5(15p+20q)}-\frac{4q(3p^2-10q-25r)^2}{(15p+20q)^2}-\frac{p(3p^2-10q-25r)^3}{(15p+20q)^3}+2pr+3qr-\frac{5(3p^2-10q-25r)r}{15p+20q}-\frac{5(3p^2-10q-25r)^2r}{(15p+20q)^2}\right)$$

Since it is a cubic polynomial in d, it is solvable, and so the fifth order equation can be brought to the form

$$y^5 + Ay + B.$$

With a little trick we can transform the equation $y^5+Ay+B=0$ to the form $z^5+z+\gamma=0$: For $A \neq 0$ we define $y =: \sqrt[4]{A}\, z$ and substitude this in the left side of the equation. We obtain $A^{\frac{5}{4}}z^5 + A^{\frac{5}{4}}z + B$, and by dividing the equation by $A^{\frac{5}{4}}$ and defining $\gamma := \frac{B}{A^{\frac{5}{4}}}$ we finally get

$$z^5 + z + \gamma = 0.$$

If we replace y by $\sqrt[4]{\frac{A}{5}}\, z$ instead of $\sqrt[4]{A}\, z$ and proceed in the same manner, we can transform the equation to

$$z^5 + 5z + \gamma = 0.$$

We will use this form in the last chapter.

Chapter 3

Klein's Solution of the Quintic

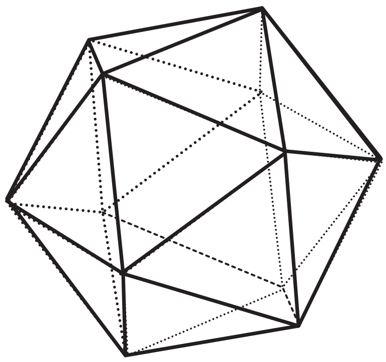
In this chapter we will describe very briefly how Klein solved the quintic equation. Klein transformed the general quintic equation to the *icosahedral equation* which can be solved by using *hypergeometric functions.* In the first two sections we will talk about the icosahedron (and follow Klein's describtion in [3]), afterwards we look on the solution of the quintic, described in [4] and [5].

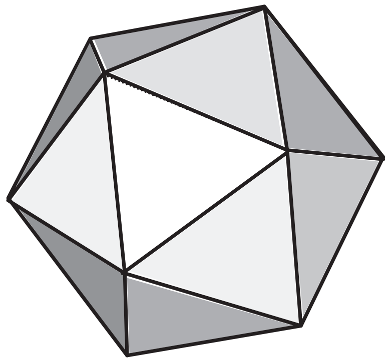
Klein,
Felix Christian,
** 25.04.1849*
in Düsseldorf/Germany
† 22.06.1925
in Göttingen/Germany

Born in 1849, Klein attended the Gymnasium in Düsseldorf and began studying mathematics and physics at the University in Bonn in 1865, at the age of 16. Only one year later he became assistant to Plücker and received his doctorate in 1868, the same year when Plücker died. After Plücker's dead he continued his "Neue Geometrie des Raumes" and got in contact with Clebsch. The famous journal "Mathematische Annalen", later managed by Klein, was originally founded by Clebsch.
In 1869, Klein first visited Berlin and afterwards Paris where he stayed till 1870 when war broke out between Prussia and France. In these two years he met Sophus Lie and Camille Jordan who introduced him to the group concept which later played an important role in his mathematical work. After doing military service Klein was appointed as lecturer at Göttingen in 1871 and as professor at Erlangen in 1872, at the age of 23!
In 1875 he moved to Munich where he was professor at the Technische Hochschule and taught students like Hurwitz and Planck. In the same year he married Anne Hegel, the granddaughter of the philosopher Georg Wilhelm Friedrich Hegel. The years 1880 to 1886 he spent at the university of Leipzig (among his students there was for example Engel), and during this years his health collapsed completely. He had to end up his career as a research mathematician and accepted a chair at the University of Göttingen. He tried to re-establish Göttingen as the most important mathematics research centre in the world, and in 1895 he brought Hilbert and his research team from Königsberg to Göttingen. With the beginning of the 20th century he became more and more interested in teaching at school level, and in 1905 he took part in formulating the "Meraner Lehrplanentwürfe". In 1913 he retired due to ill health and died in 1925. From the mathematical point of view Klein was very famous and important for his work in non-euclidean geometry, geometry and group theory, and also function theory.

3.1 The Icosahedral Group

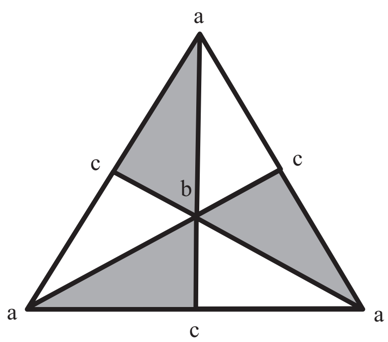
The *icosahedron* (German: Ikosaeder) consists of 20 equilateral triangles (German: gleichseitige Dreiecke) and has 12 corners. A model of such an icosahedron can be seen in the image below.

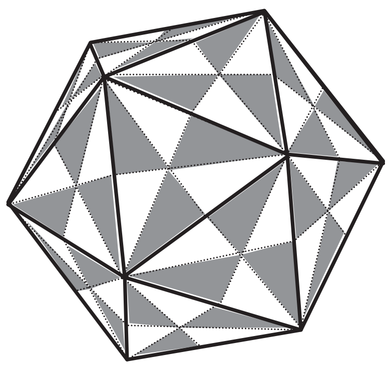




If we now imagine the icosahedron centered in a sphere and if we define the center of the sphere to be the center of the icosahedron then there are only 60 possibilities of revolving the icosahedron around its center such that it is in the same position as before:
Since we have 12 corners with always two of them on the same axis through the center we can turn them around such an axis by 5 different angels, and by revolving the icosahedron by 180° we can change the position of the two points on such an axis. This gives $\frac{12}{2} \cdot 5 \cdot 2 = 60$ possible rotations.

Take a closer look at one of the equilateral triangles which form the icosahedron. The corners of the triangle are the corners of the icosahedron, let us give them the name **a**. The middle points of the sites of the triangle get the name **c** and the middlepoint of the triangle the name **b**.





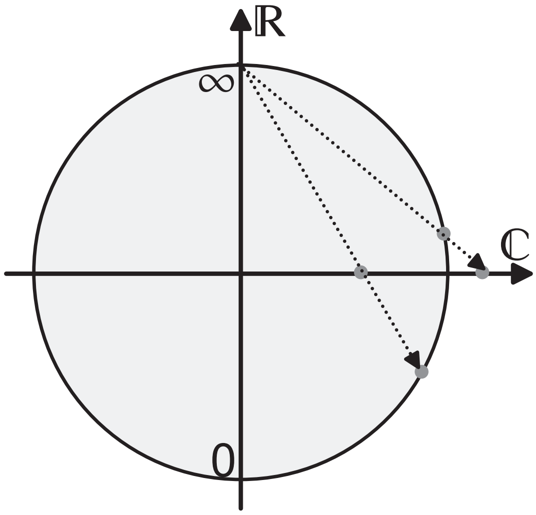
By this way we find in each triangle six smaller triangles such that always three of them are congruent (in the picture they are grey and white). On the icosahedron we have of each sort sixty congruent triangles which will be mapped onto each other by the rotations descriped above. We have twelve times a point of type **a**, twenty times one of type **b** and thirty times one of type **c**. Corner **a** touches always five triangles of each type, corner **b**

touches three triangles and corner **c** touches two triangles. If we project the triangles on the sphere around them, then corner **a** has angle $\frac{\pi}{5}$, corner **b** has angle $\frac{\pi}{3}$ and corner **c** has angle $\frac{\pi}{2}$.
The above described rotations of the icosahedron and its projection on the sphere form a group of order $60 = \frac{5!}{2}$, the *icosahedral group* G_{60}. It can be shown that this group is isomorphic to the *alternating group* A_5.

3.2 The Icosahedral Equation

Suppose a polynomial equation of degree n of the type $\varphi(z) - w\psi(z) = 0$ where φ and ψ are polynomials in the complex variable z and w is a complex parameter. Since this equation is of degree n, it has n complex roots z, depending on the parameter w, or in other words: We can classify n values of z (the roots) as belonging to one value of w. We can find these complex values on the complex *Riemann sphere.*

Now we can imagine a one-to-one mapping of the sphere on a triangle in such a way that three specific points of the sphere will be mapped on the corners of the triangle.
We have seen in the section before that we have 60 congruent triangles on the icosahedron. If Φ_{60} and Ψ_{60} are polynomials of degree 60 and $\Phi_{60}(z) - w\Psi_{60}(z) = 0$, then for every w we find 60 values of z and we imagine each of these values of z mapped in one of the triangles of the sphere, in such a way that the z for $w = 0$ is mapped on corner **b**, the z for $w = 1$ on corner **c** and the z for $w = \infty$ on corner **a**.



Now Klein changes the variables, he introduces the so-called *homogeneous variables*, i.e., he replaces w by $\frac{w_1}{w_2}$ (and z by $\frac{z_1}{z_2}$) to prevent problems with the point ∞:
Certainly $w_1, w_2 \in \mathbb{C}$, but both finite and $(w_1, w_2)^T \neq \vec{0}$. Then the point $w = \infty$ turns to $w_2 = 0$. One value of w is representated by $(\lambda w_1, \lambda w_2)^T$ with constant factor λ.
The equation

$$\Phi_{60}(z) - w\Psi_{60}(z) = 0$$

turns then to

$$\Phi_{60}(z_1, z_2) - \frac{w_1}{w_2}\Psi_{60}(z_1, z_2) = 0 \qquad \Longleftrightarrow \qquad \frac{w_1}{w_2} = \frac{\Phi_{60}(z_1, z_2)}{\Psi_{60}(z_1, z_2)}.$$

With $X_{60} := \Phi_{60} - \Psi_{60}$ we can describe the relations between w and z by the following ratio:

$$w_1 : (w_1 - w_2) : w_2 = \Phi_{60} : X_{60} : \Psi_{60}.$$

Let's sum up these relations in a little table:

$$\begin{array}{ccccccc} w=0 & \longleftrightarrow & w_1=0 & \longleftrightarrow & \Phi_{60}=0 & \longleftrightarrow & \text{corner } \mathbf{b} \\ w=1 & \longleftrightarrow & w_1-w_2=0 & \longleftrightarrow & X_{60}=0 & \longleftrightarrow & \text{corner } \mathbf{c} \\ w=\infty & \longleftrightarrow & w_2=0 & \longleftrightarrow & \Psi_{60}=0 & \longleftrightarrow & \text{corner } \mathbf{a} \end{array}$$

Since we only have $\frac{60}{3}$ corners of type **b**, $\frac{60}{2}$ corners of type **c** and $\frac{60}{5}$ corners of type **a**, the roots of $\Phi_{60},\ X_{60},\ \Psi_{60}$ must have the multiplicities 3, 2 and 5, i.e.,

$$\Phi_{60}=\varphi_{20}^3,\ X_{60}=\chi_{30}^2,\ \Psi_{60}=\psi_{12}^5.$$

How to find these equations? We will demonstrate this for ψ_{12}:
The roots of ψ_{12} are the corners of type **a** in their projection on the Riemann sphere S^2 around them so we find twelve coordinate tuples $(c,r)^T\in S^2$ $(c\in\mathbb{C},\ r\in\mathbb{R})$, and we can transform them to ordinary complex numbers by using the mapping

$$\sigma: S^2\to\bar{\mathbb{C}},\qquad \sigma(c,r):=\begin{cases}\frac{c}{1-r}, & (c,r)^T\neq(0,1)^T\\ \infty, & (c,r)^T=(0,1)^T.\end{cases}$$

So we get the following results:

c	r	$\sigma(c,r)$
0	1	∞
0	-1	0
$\frac{2}{5}\sqrt{5}e^{\frac{k\ 2\pi\imath}{5}},\ k=0,...,4$	$-\frac{1}{5}\sqrt{5}$	$\frac{1}{2}(\sqrt{5}-1)e^{\frac{k\ 2\pi\imath}{5}},\ k=0,...,4$
$\frac{2}{5}\sqrt{5}e^{\frac{(2k+1)\ 2\pi\imath}{10}},\ k=0,...,4$	$\frac{1}{5}\sqrt{5}$	$\frac{1}{2}(\sqrt{5}+1)e^{\frac{(2k+1)\ 2\pi\imath}{10}},\ k=0,...,4$

It can be checked easily that the last ten roots are roots of an equation of the form $x^{10}+11x^5-1$, or in homogeneous coordinates we can write $z_1^{10}+11z_1^5z_2^5-z_2^{10}$, the root ∞ can be described by a single factor z_2 and the root 0 by z_1, so finally we get

$$z_1z_2(z_1^{10}+11z_1^5z_2^5-z_2^{10}).$$

By a similar process we can determine the other polynomials.
Now we know the polynomials $\Phi_{60},\ X_{60},\ \Psi_{60}$ except for some constant factor, but with the relation $X_{60}=\Phi_{60}-\Psi_{60}$ we can determine them and get finally

$$\begin{aligned} w_1:(w_1-w_2):w_2 &= [-(z_1^{20}+z_2^{20})+228(z_1^{15}z_2^5-z_1^5z_2^{15})-494\ z_1^{10}z_2^{10}]^3\\ &: -[(z_1^{30}+z_2^{30})+522(z_1^{25}z_2^5-z_1^5z_2^{25})-10005(z_1^{20}z_2^{10}+z_1^{10}z_2^{20})]^2\\ &: 1728[z_1z_2(z_1^{10}+11z_1^5z_2^5-z_2^{10})]^5.\end{aligned}$$

This is the *icosahedral equation.*

3.3 The Solution of the Icosahedral Equation

From now on we choose for the *icosahedral equation* the following notation (like Klein does in [4]):

$$X = 1728 \frac{H^3(\eta_1,\eta_2)}{f^5(\eta_1,\eta_2)},$$

with $H(\eta_1,\eta_2) := \frac{1}{144}[-(\eta_1^{20}+\eta_2^{20}) + 228(\eta_1^{15}\eta_2^5 - \eta_1^5\eta_2^{15}) - 494\,\eta_1^{10}\eta_2^{10}]$,
$f(\eta_1,\eta_2) := \eta_1\eta_2(\eta_1^{10} + 11\eta_1^5\eta_2^5 - \eta_2^{10})$ and X as *parameter of the icosahedral equation.*

Finally $T(\eta_1,\eta_2) := \frac{1}{12}[(\eta_1^{30}+\eta_2^{30}) + 522(\eta_1^{25}\eta_2^5 - \eta_1^5\eta_2^{25}) - 10005(\eta_1^{20}\eta_2^{10} + \eta_1^{10}\eta_2^{20})]$ with the relation $T^2 = 12f^5 - 12^4H^3$ which can easily be verified by using mathematica for example.

A solution to the icosahedral equation can be found by using *hypergeometric functions.*

3.3.1 Definition The *hypergeometric function* ${}_pF_q(a_1,...,a_p;b_1,...,b_q;z)$ is defined by

$$ {}_pF_q(a_1,...,a_p;b_1,...,b_q;z) := \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_k\ z^k}{\prod_{j=1}^{q}(b_j)_k\ k!},$$

using the *Pochhammer symbol* $(a)_k := \prod_{n=1}^{k}(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$.
In the case $p=2$ and $q=1$ the function ${}_2F_1$ is often denoted only by F and called *Gauss series* or *Kummer series.*

3.3.2 Example Many elementary functions can be expressed by using hypergeometric functions; here are some examples using MAPLE V.4:

```
> readlib(hypergeom): hypergeom( [],[1/2],-z^2/4 );
```

$$\cos(z)$$

```
> z*hypergeom( [],[3/2],-z^2/4 );
```

$$\sin(z)$$

```
> hypergeom( [],[],z );
```

$$e^z$$

```
> z*hypergeom( [1,1],[2],-z );
```

$$\ln(1+z)$$

Klein gives in [4] one solution of the icosahedral equation $X = 1728\frac{H^3(\eta_1,\eta_2)}{f^5(\eta_1,\eta_2)}$ for $\eta_1,\ \eta_2$, using $F = {}_2F_1$ and a 12^{th} root of unity ρ:

$$\begin{cases} \eta_1 = \rho \sqrt[20]{12}\ (1-X)^{\frac{1}{60}}\ F\left(-\frac{1}{60}, \frac{29}{60}; \frac{4}{5}; \frac{1}{1-X}\right) \\ \eta_2 = \rho \frac{1}{\sqrt[20]{12^{11}}}(1-X)^{-\frac{11}{60}}\ F\left(\frac{11}{60}, \frac{41}{60}; \frac{6}{5}; \frac{1}{1-X}\right) \end{cases},\ |1-X| \geq 1,$$
$$\begin{cases} \eta_1 = \rho\ e^{-\frac{\imath\pi}{60}} \sqrt[20]{12}\ X^{\frac{1}{60}}\ F\left(-\frac{1}{60}, \frac{19}{60}; \frac{4}{5}; \frac{1}{X}\right) \\ \eta_2 = \rho\ e^{\frac{11\imath\pi}{60}} \frac{1}{\sqrt[20]{12^{11}}} X^{-\frac{11}{60}}\ F\left(\frac{11}{60}, \frac{31}{60}; \frac{6}{5}; \frac{1}{X}\right) \end{cases},\ |X| \geq 1, \text{ and otherwise}$$
$$\begin{cases} \eta_1 = \rho(0.09072 + 1.73116\imath)\ F\left(\frac{11}{60}, \frac{-1}{60}; \frac{2}{3}; X\right) + (1.02130 - 1.76895\imath)\ F\left(\frac{11}{60}, \frac{-1}{60}; \frac{1}{2}; 1-X\right) \\ \eta_2 = \rho(3.32355 - 5.11783\imath)\ F\left(\frac{11}{60}, \frac{-1}{60}; \frac{2}{3}; X\right) + (-3.01928 + 5.22954\imath)\ F\left(\frac{11}{60}, \frac{-1}{60}; \frac{1}{2}; 1-X\right). \end{cases}$$

But having one solution of the icosahedral equation, this equation is solved completly, since we can use the substitutions of the icosahedron described in section 3.1. Formulated for $\eta_1,\ \eta_2$ we get (with $\mu, \nu = 0, ..., 4$ and $\xi = e^{\frac{2\pi\imath}{5}}$):

$$\begin{array}{ll} (\pm\xi^{\frac{\nu}{2}}\eta_1,\ \ \pm\xi^{-\frac{\nu}{2}}\eta_2); & (\pm\frac{\xi^{\frac{\mu}{2}}(\xi+\xi^4)\xi^{-\frac{\nu}{2}}\eta_1+\xi^{\frac{\nu}{2}}\eta_2}{\xi^2-\xi^3},\ \ \pm\frac{\xi^{-\frac{\mu}{2}}\xi^{-\frac{\nu}{2}}\eta_1-(\xi+\xi^4)\xi^{\frac{\nu}{2}}\eta_2}{\xi^2-\xi^3}); \\ (\mp\xi^{\frac{\nu}{2}}\eta_2,\ \ \pm\xi^{-\frac{\nu}{2}}\eta_1); & (\pm\frac{\xi^{-\frac{\mu}{2}}\xi^{-\frac{\nu}{2}}\eta_1-(\xi+\xi^4)\xi^{\frac{\nu}{2}}\eta_2}{\xi^2-\xi^3},\ \ \pm\frac{\xi^{\frac{\mu}{2}}(\xi+\xi^4)\xi^{-\frac{\nu}{2}}\eta_1+\xi^{\frac{\nu}{2}}\eta_2}{\xi^2-\xi^3}). \end{array}$$

And that is the reason why Klein compares (in [5], p.483) the icosahedral equation with the equation $x^n - a$: If we found one root of $x^n - a$, we get all the roots by multiplicating it with the n-th roots of unity. Therefore Klein calls the icosahedral equation to be a "*Normalgleichung sui generis*".

3.4 The Solution of the Quintic Equation

For solving the quintic equation we have now, since we know a solution of the icosahedral equation, to proceed in the following way: (Klein [5], p.483 and p.486)

1^{st} step: Express the parameter X of the icosahedral equation by the coefficients of the quintic equation.
2^{nd} step: Solve the icosahedral equation for this parameter X.
3^{rd} step: Find a function $\eta(z_1, ..., z_5)$, depending on the roots of the quintic equation, which fulfils the substitutions of the icosahedron described above if we alternate the roots $z_1, ..., z_5$.
4^{th} step: Finally express the roots $z_1, ..., z_5$ by this η.

We will follow Klein's way described in [4]. Klein assumes the quintic equation in the form

$$y^5 + 5\alpha y^2 + 5\beta y + \gamma.$$

From chapter 1 we know that every quintic equation can be brought to this form by using Tschirnhaus transformations. To simplify the formulas Klein found, we assume furthermore $\alpha = 0$, which is always possible, since we showed in section 2.2 that a quintic equation can

be brought to this form by Bring's transformations.
At the end of section 2.2 we descriped how the quintic equation can be brought finally to the form

$$y^5 + 5y + \gamma,$$

so we also could assume $\beta = 1$.

In his article [4] Klein describes very briefly that the parameter X can be expressed as a solution of a quadratic equation whose coefficients consist only of the coefficients α, β, γ. Defining ($\alpha = 0$)

$$L := -\beta^2, \qquad M := -144\beta^5 + \gamma^4, \qquad N := -648\beta^5\gamma^2 - \gamma^6$$

and

$$l := 12^{\frac{2}{5}} \cdot L, \qquad m := 12^{-\frac{4}{3}} \cdot M, \qquad n := -\frac{1}{144} \cdot N,$$

then X is a solution of the quadratic equation

$$l^5 X^2 - (l^5 + m^3 - n^2)X + m^3 = 0.$$

So we get as (one) solution

$$\mathbf{X_1 = FullSimplify}\left[\frac{\mathbf{l^5 + m^3 - n^2 + \sqrt{(l^5 + m^3 - n^2)^2 - 4\, l^5\, m^3}}}{\mathbf{2\, l^5}}\right]$$

$$\frac{3456\,\beta^{15} + 207\,\beta^{10}\,\gamma^4 + \beta^5\,\gamma^8 - \sqrt{\beta^{10}\,\gamma^4\,(81\,\beta^5 + \gamma^4)^2\,(256\,\beta^5 + \gamma^4)}}{3456\,\beta^{15}}$$

or in general:

$$X_{1,2} = \frac{1}{3456\beta^{10}}\left((\gamma^8 + 207\beta^5\gamma^4 + 3456\beta^{10}) \mp \gamma^2(\gamma^4 + 81\beta^5)\sqrt{\gamma^4 + 256\beta^5}\right).$$

Now we will follow Klein's way of constructing a solution y_ν of $y^5 + 5\alpha y^2 + 5\beta y + \gamma = 0$, which is sometimes really hard. In his publication [4] he only gives results or very short descriptions (for more detailed information he refers to his book "Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade", which can't be found in Helsinki) and there are also many misprints in his formulas!

In the beginning we repeat some necessary basic facts:

3.4.1 Definition Suppose the polynomial $p(x) = (x - x_1)\cdots(x - x_n) =: x^n - s_1 x^{n-1} + s_2 x^{n-2} - ... + (-1)^n s_n$, then s_1, ..., s_n are called the *elementary symmetric polynomials* of $x_1, ..., x_n$ and can be written as

$$s_1 = \sum_{i_1} x_{i_1}, \quad s_2 = \sum_{i_1 < i_2} x_{i_1} x_{i_2}, \quad ..., \quad s_k = \sum_{i_1 < ... < i_k} x_{i_1} \cdots x_{i_k}, \quad ..., \quad s_n = x_1 \cdots x_n.$$

3.4.2 Lemma Suppose $p(x)$ and s_1, ..., s_n like in 3.4.1. Define

$$\sigma_k := \sum_{i=1}^{n} x_i^k, \qquad \text{for } k \geq 1.$$

Then σ_k can be expressed by the elementary symmetric polynomials by

$$\sigma_k = \begin{cases} \sum_{i=1}^{k-1} (-1)^{i+1} s_i \sigma_{k-i} + (-1)^{k+1} k s_k & \text{for } k \leq n, \\ \sum_{i=1}^{n} (-1)^{i+1} s_i \sigma_{k-i} & \text{for } k > n. \end{cases}$$

Especially we get (for n large enough)

$$\sigma_1 = s_1, \qquad \sigma_2 = s_1^2 - 2s_2 = s_1\sigma_1 - 2s_2, \qquad \sigma_3 = s_1^3 - 3s_1s_2 + 3s_3 = s_1\sigma_2 - s_2\sigma_1 + 3s_3.$$

Proof: See Tignol [8], p.38f. #

Klein looks for an expression of one root y_ν of $y^5 + 5\alpha y^2 + 5\beta y + \gamma$; in this polynomial s_1 and s_2 are equal to zero. By lemma 3.4.2 this is equivalent to $\sigma_1 = \sigma_2 = 0$, so $y_\nu(\eta_1, \eta_2)$ should be an expression where the sum over all y_ν and the sum over all y_ν^2 equals 0.
For finding such an expression he starts with (where ϵ is a fifth root of unity)

$$t_\nu := -\epsilon^\nu 5\eta_1^2\eta_2^4 + \epsilon^{2\nu}(\eta_1^6 - 2\eta_1\eta_2^5) + \epsilon^{3\nu}(\eta_2^6 + 2\eta_1^5\eta_2) - \epsilon^{4\nu}5\eta_1^4\eta_2^2, \quad ^1$$

which fulfils

$$t_0 \cdots t_4 = 12T, \qquad \sum_{\nu=0}^{4} t_\nu = 0 = \sum_{\nu=0}^{4} t_\nu^3, \qquad \sum_{\nu=0}^{4} t_\nu^2 = 20f, \qquad \sum_{\nu=0}^{4} t_\nu^4 = 20f^2.$$

These formulas can be verified easily:

```
FullSimplify[
  Expand[∏_{ν=0}^{4} (-ϵ^ν 5 η1^2 η2^4 + ϵ^(2ν) (η1^6 - 2 η1 η2^5) + ϵ^(3ν) (η2^6 + 2 η1^5 η2) - ϵ^(4ν) 5 η1^4 η2^2)]]
```

$$\eta_1^{30} + 522\,\eta_1^{25}\,\eta_2^5 - 10005\,\eta_1^{20}\,\eta_2^{10} - 10005\,\eta_1^{10}\,\eta_2^{20} - 522\,\eta_1^5\,\eta_2^{25} + \eta_2^{30}$$

$$T = \frac{1}{12}\left((\eta_1^{30} + \eta_2^{30}) + 522\,(\eta_1^{25}\,\eta_2^5 - \eta_1^5\,\eta_2^{25}) - 10005\,(\eta_1^{20}\,\eta_2^{10} + \eta_1^{10}\,\eta_2^{20})\right)$$

```
FullSimplify[Collect[
  ∑_{ν=0}^{4} (-ϵ^ν 5 η1^2 η2^4 + ϵ^(2ν) (η1^6 - 2 η1 η2^5) + ϵ^(3ν) (η2^6 + 2 η1^5 η2) - ϵ^(4ν) 5 η1^4 η2^2)^2, {η1, η2}]]
```

$$20\,\eta_1\,\eta_2\,(\eta_1^{10} + 11\,\eta_1^5\,\eta_2^5 - \eta_2^{10})$$

$$f = \eta_1\,\eta_2\,(\eta_1^{10} + 11\,\eta_2^5\,\eta_1^5 - \eta_2^{10})$$

[1] Here is the first mistake in his article: Instead of ...-$\epsilon^{4\nu}5\eta_1^4\eta_2^2$ we find there ...+$\epsilon^{4\nu}5\eta_1^4\eta_2^2$.

```
FullSimplify[Collect[
  Sum[(-ϵ^ν 5 η1^2 η2^4 + ϵ^(2ν) (η1^6 - 2 η1 η2^5) + ϵ^(3ν) (η2^6 + 2 η1^5 η2) - ϵ^(4ν) 5 η1^4 η2^2)^3, {ν, 0, 4}], {η1, η2}]]
```

0

```
FullSimplify[Collect[
  Sum[(-ϵ^ν 5 η1^2 η2^4 + ϵ^(2ν) (η1^6 - 2 η1 η2^5) + ϵ^(3ν) (η2^6 + 2 η1^5 η2) - ϵ^(4ν) 5 η1^4 η2^2)^4, {ν, 0, 4}], {η1, η2}]]
```

$$20\,\eta_1^2\,\eta_2^2\,(\eta_1^{10}+11\,\eta_1^5\,\eta_2^5-\eta_2^{10})^2$$

With 3.4.2 we find that the t_νs are solutions of the fifth order equation

$$t^5 - 10ft^3 + 45f^2t - 12T = 0.$$

But we are looking for a quintic equation without cubic term, so in a next step Klein defines $400W_\nu$ to be the *Hessian* (German: Hesse-Determinante) of t_ν, which can be written as

$$W_\nu = (\epsilon^\nu\eta_1 - \epsilon^{2\nu}\eta_2)(-\eta_1^7 + 7\eta_1^2\eta_2^5) + (\epsilon^{3\nu}\eta_1 + \epsilon^{4\nu}\eta_2)(-\eta_2^7 - 7\eta_1^5\eta_2^2).$$

```
Hesse = Expand[FullSimplify[Det[( ∂η1,η1 tν   ∂η1,η2 tν
                                  ∂η2,η1 tν   ∂η2,η2 tν )]]]
```

$$-400\,(-1)^{12\nu/5}\,\eta_1^8 + 400\,(-1)^{14\nu/5}\,\eta_1^7\,\eta_2 - 1800\,(-1)^{6\nu/5}\,\eta_1^6\,\eta_2^2 - 1000\,(-1)^{16\nu/5}\,\eta_1^6\,\eta_2^2 - 2800\,(-1)^{8\nu/5}\,\eta_1^5\,\eta_2^3 + 2800\,(-1)^{12\nu/5}\,\eta_1^3\,\eta_2^5 - 1000\,(-1)^{4\nu/5}\,\eta_1^2\,\eta_2^6 - 1800\,(-1)^{14\nu/5}\,\eta_1^2\,\eta_2^6 - 400\,(-1)^{6\nu/5}\,\eta_1\,\eta_2^7 - 400\,(-1)^{8\nu/5}\,\eta_2^8$$

```
Wν = Expand[(ϵ^ν η1 - ϵ^(2ν) η2) (-η1^7 + 7 η1^2 η2^5) + (ϵ^(3ν) η1 + ϵ^(4ν) η2) (-η2^7 - 7 η1^5 η2^2)]
```

$$-\left(e^{\frac{2i\pi}{5}}\right)^{\nu}\eta_1^8 + \left(e^{\frac{2i\pi}{5}}\right)^{2\nu}\eta_1^7\,\eta_2 - 7\left(e^{\frac{2i\pi}{5}}\right)^{3\nu}\eta_1^6\,\eta_2^2 - 7\left(e^{\frac{2i\pi}{5}}\right)^{4\nu}\eta_1^5\,\eta_2^3 + 7\left(e^{\frac{2i\pi}{5}}\right)^{\nu}\eta_1^3\,\eta_2^5 - 7\left(e^{\frac{2i\pi}{5}}\right)^{2\nu}\eta_1^2\,\eta_2^6 - \left(e^{\frac{2i\pi}{5}}\right)^{3\nu}\eta_1\,\eta_2^7 - \left(e^{\frac{2i\pi}{5}}\right)^{4\nu}\eta_2^8$$

```
FullSimplify[Hesse - 400 Wν]
```

$$-200\,(-1)^{2\nu/5}\,(-1+(-1)^{2\nu})\,\eta_1^2\,(5\,(-1)^{4\nu/5}\,\eta_1^4\,\eta_2^2 + 2\,\eta_1\,(\eta_1^5 - 7\,\eta_2^5) + (-1)^{2\nu/5}\,\eta_2\,(-2\,\eta_1^5 + 9\,\eta_2^5))$$

The last term equals zero, since $(-1 + (-1)^{2\nu}) = 0$.
Like before we can verify the following properties:

$$\sum_{\nu=0}^{4} W_\nu = 0 = \sum_{\nu=0}^{4} W_\nu^2,\ \sum_{\nu=0}^{4} W_\nu^3 = -120f^2,\ \sum_{\nu=0}^{4} W_\nu^4 = 2880fH,\ \sum_{\nu=0}^{4} W_\nu^5 = 5W_0\cdots W_4 = -5{\cdot}12^4H^2.$$

So the W_νs are solutions of the equation

$$W^5 + 40f^2W^2 - 720fHW + 12^4H^2 = 0,$$

which has no cubic and fourth order term.

Now he looks for a second term with the same properties and finally he finds

$$\sigma_\nu := 24f^2 - 7ft_\nu^2 + t_\nu^4.$$

If we now define

$$y_\nu := \lambda \frac{f}{H} W_\nu + \mu \frac{1}{f^2} \sigma_\nu,$$

then we get $\sum_{\nu=0}^4 y_\nu = 0 = \sum_{\nu=0}^4 y_\nu^2$ and he shows that these y_νs are solutions of a quintic equation of the form

$$y^5 + 5Ay^2 + 5By + \Gamma,$$

whose coefficients depend on X $(= 1728\frac{H^3}{f^5})$, λ and μ only and which is of the right form!

Before we show how to determine μ and λ (X has already been calculated), we repeat some knowledge about the discriminant:

3.4.3 Lemma Suppose the polynomial $p(x) = (x-x_1)\cdots(x-x_n) =: x^n + a_{n-1}x^{n-1} + ... + a_0$ and define the product $\pi(x_1, ..., x_n) = \prod_{1\le i<j\le n}(x_i - x_j)$. Then π^2 can be expressed by the coefficients of $p(x)$ only.

Proof: See Tignol [8], p. 106. #

3.4.4 Definition Suppose $p(x)$ and $\pi(x_1, ..., x_n)$ defined like in lemma 3.4.3. Then $D(a_0, ..., a_{n-1}) := \pi^2(x_1, ..., x_n)$ is called the *discriminant* of $p(x)$.

A way of calculating the discrimant by the coefficients of a given polynomial is described in Tignol [8], p. 112. For the polynomial $y^5 + 5\alpha y^2 + 5\beta y + \gamma$ we can use MAPLE V.4 to calculate the discriminant to

```
> p := y^5+5*a*y^2 + 5*b*y + c;
```

$$p := y^5 + 5\,a\,y^2 + 5\,b\,y + c$$

```
> D:=discrim(p,y);
```

$$D := \quad 3125\,c^4 + 281250\,a^2\,c^2\,b - 421875\,a^4\,b^2 - 1000000\,a\,b^3\,c + 800000\,b^5 + 337500\,c\,a^5$$

```
> Delta:=d/3125;
```

$$\Delta := c^4 + 90\,a^2\,c^2\,b - 135\,a^4\,b^2 - 320\,a\,b^3\,c + 256\,b^5 + 108\,c\,a^5$$

For $\alpha = 0$ we get

$$\Delta = 256\beta^5 + \gamma^4.$$ [2]

[2] Again we find a mistake in Klein's publication: Instead of $... + 256\beta^5 + ...$ he writes $... + 266\beta^5 + ...$

How to find λ and μ? Klein tries to find another expression of the factors $\frac{fW_\nu}{H}$ and $\frac{\sigma_\nu}{f^2}$ in

$$y_\nu = \lambda \frac{f}{H} W_\nu + \mu \frac{1}{f^2} \sigma_\nu,$$

and he uses the property

$$\frac{12^2 H}{W_\nu} = t_\nu^2 - 3f.$$

By defining

$$\xi_\nu := \frac{t_\nu^2}{f}$$

we get

$$\frac{fW_\nu}{H} = \frac{144}{\xi_\nu - 3} \qquad \text{and} \qquad \frac{\sigma_\nu}{f^2} = \xi_\nu^2 - 7\xi_\nu + 24.$$

In a first step he expresses ξ_ν as a function of y_ν, β, γ ($\alpha = 0$) and Δ:

$$\xi_\nu = -\frac{1}{2\beta^3} \left(-\beta\gamma y_\nu^3 + 2\beta^2 y_\nu^4 \pm \frac{1}{\sqrt{\Delta}}\, \beta y_\nu^2 (\beta + y_\nu^4)(256\beta^3 + \gamma^2 y_\nu^2 - 8\beta\gamma y_\nu^3 + 48\beta^2 y_\nu^4) \right).$$

For one of these ξ_ν we demonstrate how to find $\frac{\sigma_\nu}{f^2}$ as a polynomial in y_ν. The trick is to replace always higher potencies than four of y_ν with the help of $y_\nu^5 = -(5\beta y_\nu + \gamma)$, and so we get

```
FullSimplify[Collect[Z2 = ξv,1^2 - 7 ξv,1 + 24, yv], yv^5 == -(5 β yv + γ)]
```

$$\frac{1}{1562500\,\beta^4\,\Delta} \left(20000\,\beta^4 \left(2048\,\gamma^4 + 2800\,\gamma^2\sqrt{\Delta} + 1875\,\Delta\right) + y_\nu \left(1333\,\gamma^3 + 875\,\gamma\sqrt{\Delta} + 2\,y_\nu^5 \left(490\,\gamma^2 + 125\sqrt{\Delta} + 168\,\gamma\,y_\nu^5 + 32\,y_\nu^{10}\right)\right) \left(-2\,\gamma^2 \left(256\,\gamma^2 + 175\sqrt{\Delta}\right) + y_\nu^5 \left(309\,\gamma^3 + 175\,\gamma\sqrt{\Delta} + 4\,y_\nu^5 \left(117\,\gamma^2 - 25\sqrt{\Delta} + 84\,\gamma\,y_\nu^5 + 16\,y_\nu^{10}\right.\right.\right.$$

```
ExpandAll[
 ReplaceRepeated[1/(1562500 β^4 Δ) (20000 β^4 (2048 γ^4 + 2800 γ^2 √Δ + 1875 Δ) + yv (1333 γ^3 +
      875 γ √Δ + 2 yv^5 (490 γ^2 + 125 √Δ + 168 γ yv^5 + 32 yv^10)) (-2 γ^2 (256 γ^2 + 175 √Δ) +
      yv^5 (309 γ^3 + 175 γ √Δ + 4 yv^5 (117 γ^2 - 25 √Δ + 84 γ yv^5 + 16 yv^10)))),
  {yv^5 -> -(5 β yv + γ), yv^10 → (5 β yv + γ)^2}]]
```

$$24 + \frac{16384\,\gamma^4}{625\,\Delta} + \frac{896\,\gamma^2}{25\sqrt{\Delta}} - \frac{\gamma^3\,y_\nu}{4\,\beta^4} - \frac{\gamma^7\,y_\nu}{4\,\beta^4\,\Delta} - \frac{\gamma^5\,y_\nu}{2\,\beta^4\sqrt{\Delta}} - \frac{\gamma^2\,y_\nu^2}{4\,\beta^3} +$$
$$\frac{3\,\gamma^6\,y_\nu^2}{4\,\beta^3\,\Delta} + \frac{\gamma^4\,y_\nu^2}{2\,\beta^3\sqrt{\Delta}} + \frac{\gamma\,y_\nu^3}{2\,\beta^2} - \frac{2\,\gamma^5\,y_\nu^3}{\beta^2\,\Delta} - \frac{\gamma^3\,y_\nu^3}{2\,\beta^2\sqrt{\Delta}} + \frac{2\,y_\nu^4}{\beta} + \frac{4\,\gamma^4\,y_\nu^4}{\beta\,\Delta} + \frac{2\,\gamma^2\,y_\nu^4}{\beta\sqrt{\Delta}} +$$
$$\frac{16384\,\gamma^3\,y_\nu^5}{625\,\Delta} + \frac{696\,\gamma\,y_\nu^5}{25\sqrt{\Delta}} - \frac{8384\,\beta\,\gamma^2\,y_\nu^6}{125\,\Delta} - \frac{96\,\beta\,y_\nu^6}{5\sqrt{\Delta}} + \frac{3584\,\beta^2\,\gamma\,y_\nu^7}{25\,\Delta} - \frac{1024\,\beta^3\,y_\nu^8}{5\,\Delta}$$

$$\mathbf{Collect}\Big[\mathbf{ReplaceRepeated}\Big[24+\frac{16384\,\gamma^4}{625\,\Delta}+\frac{896\,\gamma^2}{25\,\sqrt{\Delta}}-\frac{\gamma^3\,y_\nu}{4\,\beta^4}-\frac{\gamma^7\,y_\nu}{4\,\beta^4\,\Delta}-\frac{\gamma^5\,y_\nu}{2\,\beta^4\,\sqrt{\Delta}}-\frac{\gamma^2\,y_\nu^2}{4\,\beta^3}+\frac{3\,\gamma^6\,y_\nu^2}{4\,\beta^3\,\Delta}+\frac{\gamma^4\,y_\nu^2}{2\,\beta^3\,\sqrt{\Delta}}+\frac{\gamma\,y_\nu^3}{2\,\beta^2}-\frac{2\,\gamma^5\,y_\nu^3}{\beta^2\,\Delta}-\frac{\gamma^3\,y_\nu^3}{2\,\beta^2\,\sqrt{\Delta}}+\frac{2\,y_\nu^4}{\beta}+\frac{4\,\gamma^4\,y_\nu^4}{\beta\,\Delta}+\frac{2\,\gamma^2\,y_\nu^4}{\beta\,\sqrt{\Delta}}+\frac{16384\,\gamma^3\,y_\nu^5}{625\,\Delta}-\frac{696\,\gamma\,y_\nu^5}{25\,\sqrt{\Delta}}-\frac{8384\,\beta\,\gamma^2\,y_\nu^6}{125\,\Delta}-\frac{96\,\beta\,y_\nu^6}{5\,\sqrt{\Delta}}+\frac{3584\,\beta^2\,\gamma\,y_\nu^7}{25\,\Delta}-\frac{1024\,\beta^3\,y_\nu^8}{5\,\Delta},\ \{y_\nu{}^5\ \text{->}\ -(5\,\beta\,y_\nu+\gamma),\ y_\nu{}^6\to-(5\,\beta\,y_\nu+\gamma)\,y_\nu,\ y_\nu{}^7\ \text{->}\ -(5\,\beta\,y_\nu+\gamma)\,y_\nu^2,\ y_\nu{}^8\ \text{->}\ -(5\,\beta\,y_\nu+\gamma)\,y_\nu^3\}\Big],\ y_\nu\Big]$$

$$24+\frac{8\,\gamma^2}{\sqrt{\Delta}}+\left(-\frac{\gamma^3}{2\,\beta^4}-\frac{120\,\beta\,\gamma}{\sqrt{\Delta}}-\frac{\gamma^5}{2\,\beta^4\,\sqrt{\Delta}}\right)y_\nu+\left(\frac{\gamma^2}{2\,\beta^3}+\frac{96\,\beta^2}{\sqrt{\Delta}}+\frac{\gamma^4}{2\,\beta^3\,\sqrt{\Delta}}\right)y_\nu^2+\left(-\frac{3\,\gamma}{2\,\beta^2}-\frac{\gamma^3}{2\,\beta^2\,\sqrt{\Delta}}\right)y_\nu^3+\left(\frac{6}{\beta}+\frac{2\,\gamma^2}{\beta\,\sqrt{\Delta}}\right)y_\nu^4$$

.

Proceeding a similar way we find for $\frac{fW_\nu}{H}$ $(=:Z_1)$

$$Z_1=\mathbf{Collect}\Big[\mathbf{Expand}\Big[-\frac{72}{M\,\sqrt{\Delta}}\Big(\big(160\,\beta^5\,\gamma^2-\sqrt{\Delta}\,(96\,\beta^5)\big)+\big(-96\,\beta^6\,\gamma-\beta\,\gamma^5-\sqrt{\Delta}\,(-\beta\,\gamma^3)\big)\,y_\nu+\big(-384\,\beta^7+\beta^2\,\gamma^4-\sqrt{\Delta}\,(-\beta^2\,\gamma^2)\big)\,y_\nu^2+\big(-10\,\beta^3\,\gamma^3-\sqrt{\Delta}\,(-6\,\beta^3\,\gamma)\big)\,y_\nu^3+\big(40\,\beta^4\,\gamma^2-\sqrt{\Delta}\,(24\,\beta^4)\big)\,y_\nu^4\Big)\Big],\ y_\nu\Big]$$

$$\frac{6912\,\beta^5}{-144\,\beta^5+\gamma^4}-\frac{11520\,\beta^5\,\gamma^2}{(-144\,\beta^5+\gamma^4)\,\sqrt{\Delta}}+\left(-\frac{72\,\beta\,\gamma^3}{-144\,\beta^5+\gamma^4}+\frac{6912\,\beta^6\,\gamma}{(-144\,\beta^5+\gamma^4)\,\sqrt{\Delta}}+\frac{72\,\beta\,\gamma^5}{(-144\,\beta^5+\gamma^4)\,\sqrt{\Delta}}\right)y_\nu+\left(-\frac{72\,\beta^2\,\gamma^2}{-144\,\beta^5+\gamma^4}+\frac{27648\,\beta^7}{(-144\,\beta^5+\gamma^4)\,\sqrt{\Delta}}-\frac{72\,\beta^2\,\gamma^4}{(-144\,\beta^5+\gamma^4)\,\sqrt{\Delta}}\right)y_\nu^2+\left(-\frac{432\,\beta^3\,\gamma}{-144\,\beta^5+\gamma^4}+\frac{720\,\beta^3\,\gamma^3}{(-144\,\beta^5+\gamma^4)\,\sqrt{\Delta}}\right)y_\nu^3+\left(\frac{1728\,\beta^4}{-144\,\beta^5+\gamma^4}-\frac{2880\,\beta^4\,\gamma^2}{(-144\,\beta^5+\gamma^4)\,\sqrt{\Delta}}\right)y_\nu^4$$

.

If we write $\frac{fW_\nu}{H}=A_4y_\nu^4+\ldots+A_0$ and $\frac{\sigma_\nu}{f^2}=B_4y_\nu^4+\ldots+B_0$ and compare the coefficients, then we get that $\frac{A_0}{B_0}=\frac{A_3}{B_3}=\frac{A_4}{B_4}$ (below only for $\frac{A_0}{B_0}$):

$$\mathbf{Coefficient0}=\mathbf{FullSimplify}\left[\frac{\frac{6912\,\beta^5}{-144\,\beta^5+\gamma^4}-\frac{11520\,\beta^5\,\gamma^2}{(-144\,\beta^5+\gamma^4)\,\sqrt{\Delta}}}{24+\frac{8\,\gamma^2}{\sqrt{\Delta}}}\right]$$

$$\frac{288\,\beta^5\,\left(5\,\gamma^2-3\,\sqrt{\Delta}\right)}{(144\,\beta^5-\gamma^4)\,\left(\gamma^2+3\,\sqrt{\Delta}\right)}$$

.

The expression for $\frac{A_2}{B_2}$ looks first different,

$$\frac{144\,\beta^5\,\left(-384\,\beta^5+\gamma^4+\gamma^2\,\sqrt{\Delta}\right)}{(144\,\beta^5-\gamma^4)\,\left(192\,\beta^5+\gamma^4+\gamma^2\,\sqrt{\Delta}\right)}$$

,

but with $\Delta=256\beta^5+\gamma^4$ we get

$$\frac{A_2}{B_2} = \frac{288\beta^5}{(144\beta^5 - \gamma^4)} \cdot \frac{-2(192\beta^5 + \gamma^4 + \gamma^2\sqrt{\Delta}) + 3\gamma^2(\gamma^2 + \sqrt{\Delta})}{2(192\beta^5 + \gamma^4 + \gamma^2\sqrt{\Delta})}$$

$$= \frac{288\beta^5}{(144\beta^5 - \gamma^4)} \cdot \Big(-1 + \frac{3\gamma^2(\gamma^2 + \sqrt{\Delta})}{\underbrace{2(\frac{3}{4}(256\beta^5 + \gamma^4) + \frac{1}{4}\gamma^4 + \gamma^2\sqrt{\Delta})}_{=\frac{1}{4}(\gamma^2+\sqrt{\Delta})(\gamma^2+3\sqrt{\Delta})}}\Big) = \frac{288\beta^5(5\gamma^2 - 3\sqrt{\Delta})}{(144\beta^5 - \gamma^4)(\gamma^2 + 3\sqrt{\Delta})} =: C.$$

In the same way we can simplify the expression of $\frac{A_1}{B_1}$,

$$\frac{144\,\beta^5\,\left(96\,\beta^5 + \gamma^4 - \gamma^2\,\sqrt{\Delta}\right)}{(144\,\beta^5 - \gamma^4)\,\left(240\,\beta^5 + \gamma^4 + \gamma^2\,\sqrt{\Delta}\right)},$$

to

$$C_1 := \frac{288\beta^5(5\gamma^2 - 3\sqrt{\Delta})}{(144\beta^5 - \gamma^4)} \cdot \frac{(\gamma^2 - \sqrt{\Delta})}{5(\gamma^2 + \sqrt{\Delta})(\gamma^2 + 15\sqrt{\Delta})}.$$

Now we can write $A_k = CB_k$ for $k = 0, 2, 3, 4$ and $A_1 = C_1B_1$ and get finally

$$y_\nu = \lambda\frac{f}{H}W_\nu + \mu\frac{1}{f^2}\sigma_\nu \stackrel{!}{=} (\lambda C_1 + \mu)B_1\ y_\nu + \sum_{i\neq 1}(\lambda C + \mu)B_i\ y_\nu^i,$$

so we have to solve the equations $(\lambda C_1 + \mu)B_1 = 1$ and $(\lambda C + \mu) = 0$ for μ and λ. From the second equation we obtain $\mu = -\lambda C$ and substituding this in the first equation we get $\lambda B_1(C_1 - C) = 1$, which can be solved to

$$\boldsymbol{\lambda = \mathtt{FullSimplify}\Bigg[\left(\left(-\frac{\gamma^3}{2\,\beta^4} - \frac{120\,\beta\,\gamma}{\sqrt{\Delta}} - \frac{\gamma^5}{2\,\beta^4\,\sqrt{\Delta}}\right)\left(\frac{288\,\beta^5\,\left(5\,\gamma^2 - 3\,\sqrt{\Delta}\right)}{(144\,\beta^5 - \gamma^4)}\,\frac{\left(\gamma^2 - \sqrt{\Delta}\right)}{5\,\left(\gamma^2 + \sqrt{\Delta}\right)\,\left(\gamma^2 + 15\,\sqrt{\Delta}\right)} - \frac{288\,\beta^5\,\left(5\,\gamma^2 - 3\,\sqrt{\Delta}\right)}{(144\,\beta^5 - \gamma^4)\,\left(\gamma^2 + 3\,\sqrt{\Delta}\right)}\right)\right)\mathtt{\hat{}}(-1)\Bigg]}$$

$$\frac{5\,(144\,\beta^5 - \gamma^4)\,\left(\gamma^2 + \sqrt{\Delta}\right)\,\left(\gamma^2 + 3\,\sqrt{\Delta}\right)\,\left(\gamma^2 + 15\,\sqrt{\Delta}\right)\,\sqrt{\Delta}}{288\,\beta\,\left(240\,\beta^5\,\gamma + \gamma^5 + \gamma^3\,\sqrt{\Delta}\right)\,\left(10\,\gamma^6 + 189\,\gamma^4\,\sqrt{\Delta} + 78\,\gamma^2\,\Delta - 117\,\Delta^{3/2}\right)}$$

$$\boldsymbol{\lambda = \frac{5\,(144\,\beta^5 - \gamma^4)\,\left(\gamma^2 + 3\,\sqrt{\Delta}\right)\,\sqrt{\Delta}}{18\,\beta\,\gamma\,\left(2\,\gamma^4 + 39\,\gamma^2\,\sqrt{\Delta} + 39\,\Delta\right)\,\left(5\,\gamma^2 - 3\,\sqrt{\Delta}\right)}}$$

$$\boldsymbol{\mu = \mathtt{FullSimplify}\Big[-\lambda\,\frac{288\,\beta^5\,\left(5\,\gamma^2 - 3\,\sqrt{\Delta}\right)}{(144\,\beta^5 - \gamma^4)\,\left(\gamma^2 + 3\,\sqrt{\Delta}\right)}\Big]}$$

$$-\frac{80\,\beta^4\,\sqrt{\Delta}}{2\,\gamma^5 + 39\,\gamma^3\,\sqrt{\Delta} + 39\,\gamma\,\Delta}.$$

Now the solution of the quintic is determined completely!

3.4.5 Example Let us use Klein's method to solve the quintic equation $y^5 + 5y - 6 = 0$, but before that we will analyze this equation 'traditionally'. Obviously $y = 1$ is a solution, so we may divide the equation by $(y-1)$ and solve the resulting fourth order equation:

```
FullSimplify[(y^5 + 5 y - 6) / (y - 1)]     6 + y + y^2 + y^3 + y^4
```

```
N[Solve[6 + y + y^2 + y^3 + y^4 == 0, y]]

{{y → 0.774891 - 1.231 i}, {y → 0.774891 + 1.231 i},
 {y → -1.27489 - 1.1002 i}, {y → -1.27489 + 1.1002 i}}
```

Now we start solving the equation by using Klein's ideas. In the notation of Klein we have $\beta = 1$ and $\gamma = -6$ and we can calculate the expressions L, M, N, l, m, n. After that we substitude the last three of these values in the formulas for calculating the parameter of the icosahedral equation X:

$$X_1 = \text{FullSimplify}\left[\frac{l^5 + m^3 - n^2 + \sqrt{(l^5 + m^3 - n^2)^2 - 4\, l^5\, m^3}}{2\, l^5}\right] \qquad \frac{1}{8}\left(4517 - 459\sqrt{97}\right)$$

$$X_2 = \text{FullSimplify}\left[\frac{l^5 + m^3 - n^2 - \sqrt{(l^5 + m^3 - n^2)^2 - 4\, l^5\, m^3}}{2\, l^5}\right] \qquad \frac{1}{8}\left(4517 + 459\sqrt{97}\right)$$

Approximately we have $X_1 \approx -0.453216$ and $X_2 \approx 1129.7$, so obviously $|X_2| \geq 0$ and we can calculate η_1, η_2 by

$$\eta_1 = \text{N}\left[e^{-\frac{i\pi}{60}}\sqrt[20]{12}\, X_2^{\frac{1}{60}}\, \text{Hypergeometric2F1}\left[-\frac{1}{60}, \frac{19}{60}, \frac{4}{5}, \frac{1}{X_2}\right]\right]$$

```
1.27129 - 0.0666253 i
```

$$\eta_2 = \text{N}\left[e^{\frac{11 i\pi}{60}}\frac{1}{\sqrt[20]{12^{11}}}\, X_2^{-\frac{11}{60}}\, \text{Hypergeometric2F1}\left[\frac{11}{60}, \frac{31}{60}, \frac{6}{5}, \frac{1}{X_2}\right]\right]$$

```
0.0589335 + 0.0382718 i
```

We will use in the whole process only approximated values since they take less place. Next we calculate t_ν, W_ν, f, H and σ_ν: (Note that the value of f should be exactly 1!)

$$t_\nu = \text{N}[-\epsilon^\nu\, 5\, \eta_1^2\, \eta_2^4 + \epsilon^{2\nu}\, (\eta_1^6 - 2\, \eta_1\, \eta_2^5) + \epsilon^{3\nu}\, (\eta_2^6 + 2\, \eta_2\, \eta_1^5) - \epsilon^{4\nu}\, 5\, \eta_1^4\, \eta_2^2]$$

```
(0.000116131 - 0.000159841 i) (0.309017 + 0.951057 i)^ν +
 (4.04802 - 1.31528 i) (0.309017 + 0.951057 i)^(2.ν) +
 (0.446893 + 0.145204 i) (0.309017 + 0.951057 i)^(3.ν) -
 (0.0381142 + 0.0524597 i) (0.309017 + 0.951057 i)^(4.ν)
```

$$W_\nu = N[(\epsilon^\nu \eta_1 - \epsilon^{2\nu} \eta_2)(-\eta_1^7 + 7\eta_1^2 \eta_2^5) + (\epsilon^{3\nu}\eta_1 + \epsilon^{4\nu}\eta_2)(-\eta_2^7 - 7\eta_1^5 \eta_2^2)]$$

$$(-5.05857 + 1.9418\, i)\,((1.27129 - 0.0666253\, i)(0.309017 + 0.951057\, i)^{\nu} - (0.0589335 + 0.0382718\, i)(0.309017 + 0.951057\, i)^{2.\nu}) - (0.0727289 + 0.0898128\, i)\,((1.27129 - 0.0666253\, i)(0.309017 + 0.951057\, i)^{3.\nu} + (0.0589335 + 0.0382718\, i)(0.309017 + 0.951057\, i)^{4.\nu})$$

$$f = N[\eta_1 \eta_2 (\eta_1^{10} + 11\eta_2^5 \eta_1^5 - \eta_2^{10})] \qquad 1. - 1.66533 \times 10^{-16}\, i$$

$$H = N\left[\frac{1}{12^2}\left(-(\eta_1^{20} + \eta_2^{20}) + 228(\eta_1^{15}\eta_2^5 - \eta_1^5 \eta_2^{15}) - 494\,\eta_1^{10}\eta_2^{10}\right)\right] \qquad -0.433954 + 0.75163\, i$$

$$\sigma_\nu = N[24 f^2 - 7 f t_\nu^2 + t_\nu^4]$$

$$(24. - 7.99361 \times 10^{-15}\, i) - (7. - 1.16573 \times 10^{-15}\, i)\,((0.000116131 - 0.000159841\, i)(0.309017 + 0.951057\, i)^{\nu} + (4.04802 - 1.31528\, i)(0.309017 + 0.951057\, i)^{2.\nu} + (0.446893 + 0.145204\, i)(0.309017 + 0.951057\, i)^{3.\nu} - (0.0381142 + 0.0524597\, i)(0.309017 + 0.951057\, i)^{4.\nu})\text{^}2 + ((0.000116131 - 0.000159841\, i)(0.309017 + 0.951057\, i)^{\nu} + (4.04802 - 1.31528\, i)(0.309017 + 0.951057\, i)^{2.\nu} + (0.446893 + 0.145204\, i)(0.309017 + 0.951057\, i)^{3.\nu} - (0.0381142 + 0.0524597\, i)(0.309017 + 0.951057\, i)^{4.\nu})\text{^}4$$

Now we need the parameters λ and μ, and with $\Delta = 256\beta^5 + \gamma^4 = 1552$ we obtain

$$\lambda = \frac{5\,(144\beta^5 - \gamma^4)\left(\gamma^2 + 3\sqrt{\Delta}\right)\sqrt{\Delta}}{18\beta\gamma\left(2\gamma^4 + 39\gamma^2\sqrt{\Delta} + 39\Delta\right)\left(5\gamma^2 - 3\sqrt{\Delta}\right)} \qquad \mu = -\frac{80\beta^4\sqrt{\Delta}}{2\gamma^5 + 39\gamma^3\sqrt{\Delta} + 39\gamma\Delta}$$

$$\frac{640\sqrt{97}\left(36 + 12\sqrt{97}\right)}{3\left(180 - 12\sqrt{97}\right)\left(63120 + 5616\sqrt{97}\right)} \qquad -\frac{320\sqrt{97}}{-378720 - 33696\sqrt{97}}.$$

So finally we get the five solutions $y_0, ..., y_4$ of the given quintic equation:

$$y_\nu = N\left[\lambda \frac{f W_\nu}{H} + \mu \frac{\sigma_\nu}{f^2}\right]$$

y_0 = ReplaceAll[y_ν, $\nu \to 0$]

$0.781741 - 1.24188\, i$

y_1 = ReplaceAll[y_ν, $\nu \to 1$]

$-1.28616 + 1.10993\, i$

y_2 = ReplaceAll[y_ν, $\nu \to 2$]

$1.00884 - 4.24498 \times 10^{-17}\, i$

y_3 = ReplaceAll[y_ν, $\nu \to 3$]

$-1.28616 - 1.10993\, i$

y_4 = ReplaceAll[y_ν, $\nu \to 4$]

$0.781741 + 1.24188\, i$.

These values coincide roughly with the 'exact' values calculated in the beginning, but we should remember that we always used the approximated results during the above process.

Bibliography

[1] Moritz Cantor: *Vorlesungen über Geschichte der Mathematik - Dritter Band.* B. G. Teubner Verlag, Leipzig (1901), p.112-118.

[2] Moritz Cantor: *Vorlesungen über Geschichte der Mathematik - Vierter Band.* B. G. Teubner Verlag, Leipzig (1908), p.130-132.

[3] Felix Klein: *Elementarmathematik vom höheren Standpunkt aus I.* Dritte Auflage, Springer-Verlag, Berlin (1924).

[4] Felix Klein: *Gesammelte Mathematische Abhandlungen - Zweiter Band.* Springer-Verlag, Berlin (1922), p. 321-380.

[5] Felix Klein: *Gesammelte Mathematische Abhandlungen - Zweiter Band.* Springer-Verlag, Berlin (1922), p. 481-502.

[6] M. Kracht and E. Kreyszig: *E.W. von Tschirnhaus: His Role in Early Calculus and his Work and Impact on Algebra.* in: Historia Mathematica Vol. 17 No. 1, Academic Press, INC, 1990, p. 16-36.

[7] Wolfgang Pfeifer: *Ethymologisches Wörterbuch des Deutschen.* 5. Auflage, Deutscher Taschenbuch Verlag, München (2000).

[8] Jean-Pierre Tignol: *Galois' Theory of Algebraic Equations.* World Scientific Publishing, Singapore, New Jersey, London, Hong Kong (2001).

[9] B.L. van der Waerden: *Algebra I.* Vierte Auflage, Springer-Verlag, Berlin, Göttingen, Heidelberg (1955).

www-libraries

[10] *Biographisch-Bibliographisches Kirchenlexikon.* Verlag Traugott Bautz (1997). http://www.bautz.de/bbkl

[11] *The MacTutor History of Mathematics archive.* http://www-history.mcs.st-andrews.ac.uk

[12] *Wolfram Research.* http://library.wolfram.com